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CAPACITORS AND RESISTANCE-CAPACITANCE NETWORKS.

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This programed textbook was developed under a contract with the United States Office of Education as Number 5 in a series of materials for use in an electrical engineering sequence. It is divided into three parts--(1) capacitors, (2) voltage-current relationships, and (3) simple resistance-capacitance networks. (DH)

No. 5

CAPACITORS AND

RESISTANCE-CAPACITANCE NETWORKS

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Contract No. OE 4-10-102
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2^{nd} Edition

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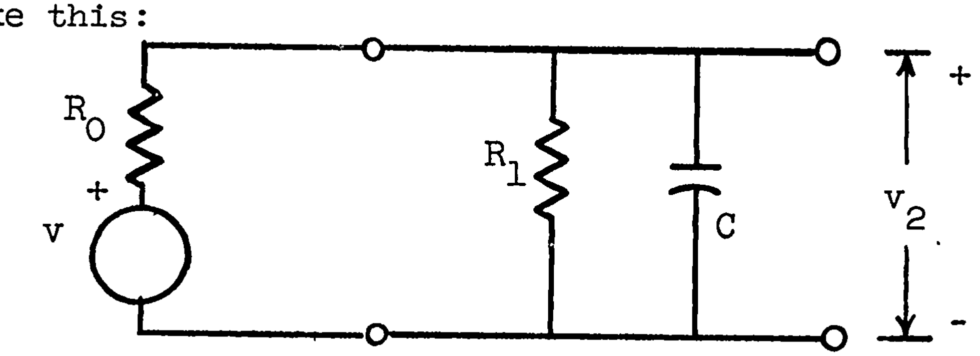
CAPACITORS AND RC NETWORKS

This unit of programmed text is concerned with a determination of voltages or currents (network variables) in a network containing resistors, sources and capacitors. What we want to do is to be able to apply Kvl, Kcl and variations of Ohm's law to networks like this:



Such networks are very much like the ones with which we have been working, except for branches containing capacitors.

As soon as we can write expressions for the currents and voltages associated with capacitors, we can treat this network in the same way that we handled purely resistive networks.

When you finish this program you will be able:

1. to determine the values of voltage and current in network branches containing capacitors when the voltages and currents in other appropriate parts of the network are known at a specific time, by applying the basic relationships: Kvl, Kcl and the v-i relationships of the branch.
2. to do number 1, above, when the known currents and voltages are functions of time.
3. to write the differential equations that result from an application of the basic relationships to a network of moderate complexity which contains resistors, capacitors and sources.
4. to solve the above differential equations when the network contains one capacitor, one source and one or more resistors. The voltage or current source may be (a) constant, (b) an exponential function, (c) a polynomial in time, or (d) a sinusoid.
5. to describe, mathematically and graphically, the response of single RC networks to switching changes of current or voltage.

1. Capacitors

A capacitor is nothing more than a device consisting of two conducting bodies which carry equal and opposite electrical charges. The tuning capacitor used in a radio for selecting stations, and shown in Fig. 1, is a familiar example of a capacitor.

Considered as a whole, the net charge on the whole capacitor equals ________________.

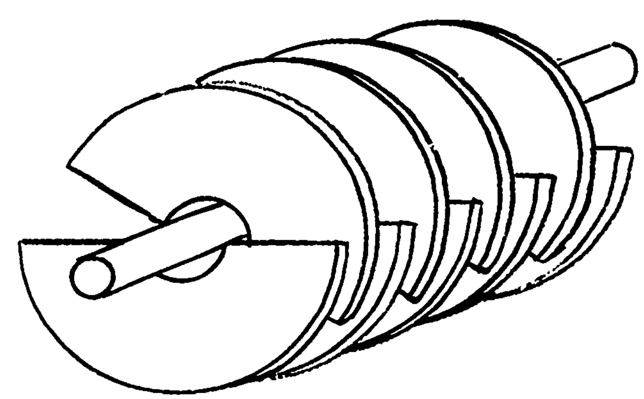


Fig. 1

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Answer: zero.

(A charge on one of two bodies, or plates, is always balanced by an equal and "opposite" charge on the other plate, i.e., their algebraic sum is always zero.)

Even though there is no net charge on a capacitor, the fact that there is a positive charge on one part of the capacitor and a negative charge on the other, and these two parts are separated from each other, means that there will be an electric field between the two parts and therefore a potential difference, or voltage across the capacitor.

Let q refer to the positive charge on one part of the capacitor and let v be the voltage between the two parts, with reference plus at the positively charged part. We call q the charge on the capacitor, or the capacitor charge, even though it is the charge on only one part. It is empirically found that these two quantities the voltage and the charge are proportional to each other. Hence, we can write

$$q = Cv$$

where C is the constant of proportionality; it is called the capacitance. As evident from this relationship, capacitance has the units of ________________.

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Answer:

capacitance has the units of coulombs/volt.

In honor of Michael Faraday, the English scientist, this unit (coulombs/volt) is called the <u>farad</u>.

The voltage across the two parts of a capacitor is 10 volts and on one of the parts there are a trillion, 10^{12}, excess electrons. Find the capacitance in <u>microfarads</u>, abbreviated μf. (The mass of an electron is 9×10^{-31} kg and its charge is -1.6×10^{-19} coulombs.)

C = ____________________

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<u>Answer</u>:

$$C = \frac{q}{V} = \frac{10^{12}(1.6 \times 10^{-19})}{10} \text{ farads} = \underline{\quad\quad} \mu f.$$

(Note that q is the <u>positive</u> charge, which is equal, but opposite, to the negative charge of the electrons.)

Remark

In general, a capacitor may be considered to be _any_ two conducting bodies carrying equal and opposite charges, as shown in Fig. 2(a). In almost any electrical device one can think of, there are conducting parts which carry charges and are at different potentials. These parts, then, can be considered to _be_ capacitors and the influence of their capacitances should be taken into account in analyzing the characteristics of the electrical device. Sometimes an engineer will take considerable pains to eliminate "stray" capacitance effects. Other times, the engineer will deliberately introduce capacitors into a device or system in order to obtain desired results.

As you learn about the functions which capacitors perform, you will be able to analyze the performance of networks involving capacitors, and design circuits to obtain the characteristics you want. Just as resistors can be manufactured in many sizes (not just physical sizes but values of R), so also capacitors can be manufactured in many sizes, covering the range of values from about 100 micromicrofarads (10^{-12} μf) to 100 microfarads (100 μf), and in many sizes and shapes. (You will learn more about the evaluation of the capacitance of capacitors having different geometries when you study electric fields, later.)

One of the most practical shapes is the parallel-plate capacitor, Fig. 2(b), in which the two conducting parts are plane surfaces separated by a small distance. (The radio tuning capacitor has such a structure with a number of parallel plates.) In an effort to increase the area of the plates, which is a factor in determining the capacitance value, many capacitors are made of two sheets of conducting foil separated by an insulating sheet, the whole thing being rolled into a compact cylindrical shape.

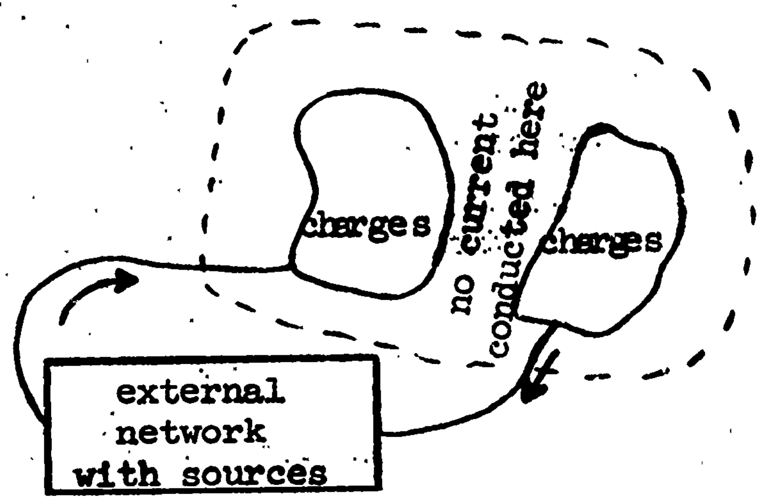
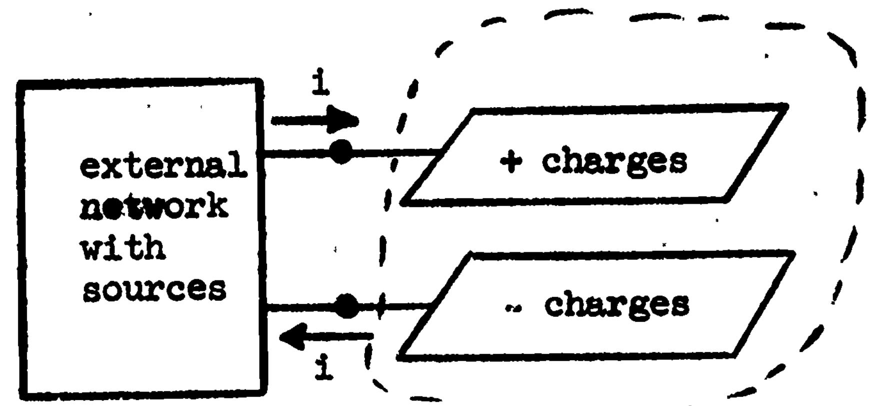


Fig. 2(a)



PARALLEL PLATE CAPACITOR

Fig. 2(b)

2. Voltage-Current Relationships in Capacitors

Consider the situation in Fig. 2(b) where two conducting bodies are connected to a network containing sources with time-varying voltages or currents. The charges on the two "plates" will vary with time. Charge will be conducted across from one plate to the other (assuming an ideal nonconductor between them).

If we don't concern ourselves with anything internal to the dashed lines in the diagram and look at the device only from its terminals, it will appear as if there is a current entering at one terminal and being conducted through to the other terminal. There will also appear a voltage across these terminals.

The symbol for a capacitor is shown in Fig. 3. A single current is shown as if it were being conducted straight through. Remembering that $q = Cv$ for the capacitor, write an expression relating the current i to the voltage.

i = ____________________

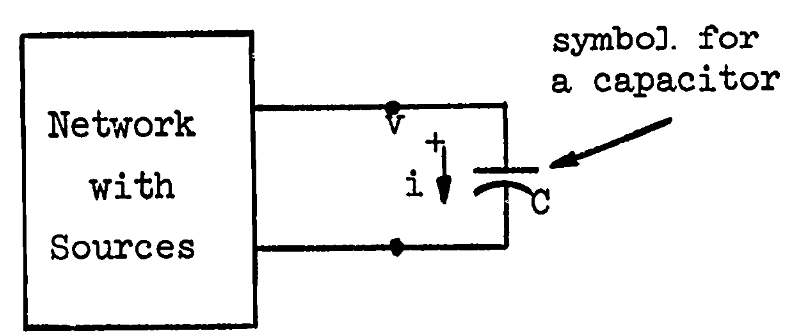


Fig. 3

<u>Answer</u>:

$\underline{i = C\frac{dv}{dt}}$ (since $i = \frac{dq}{dt}$)

(To emphasize that these are functions of time, we sometimes write $i(t) = C\frac{dv(t)}{dt}$.)

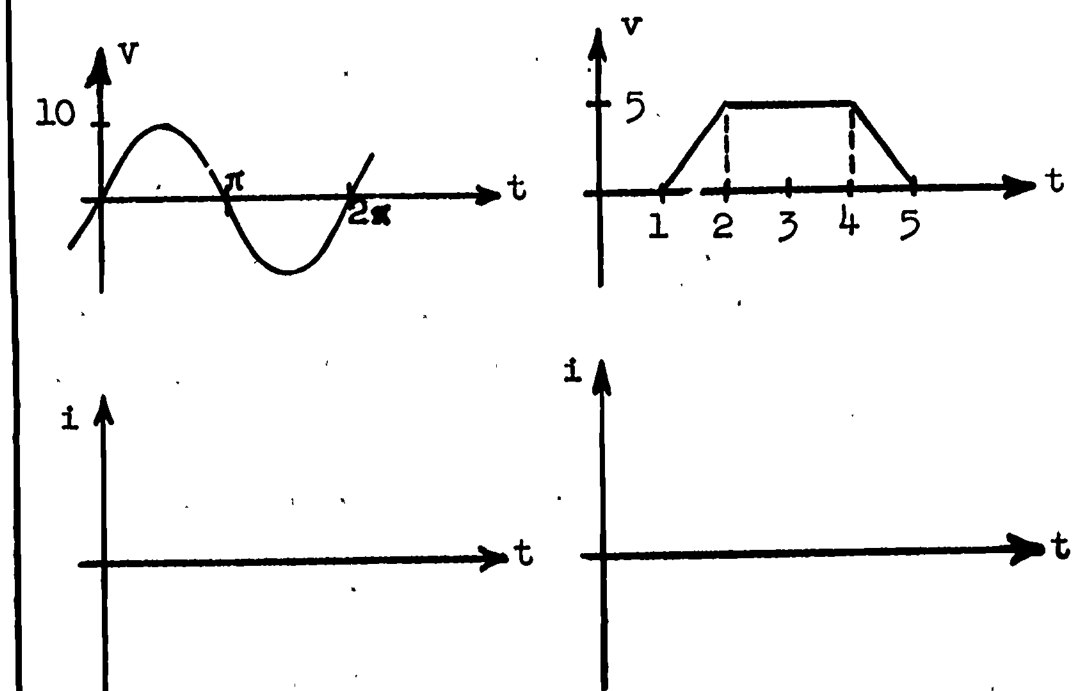


Fig. 4

This expression, $i = C \frac{dv}{dt}$, is the current-voltage relationship for a capacitor. In importance, it ranks with Ohm's law which gives the analogous relationship for a resistor.

Figure 4 shows some possible capacitor voltage waveforms. Sketch on the axes below each waveform, the corresponding current waveforms and label them appropriately.

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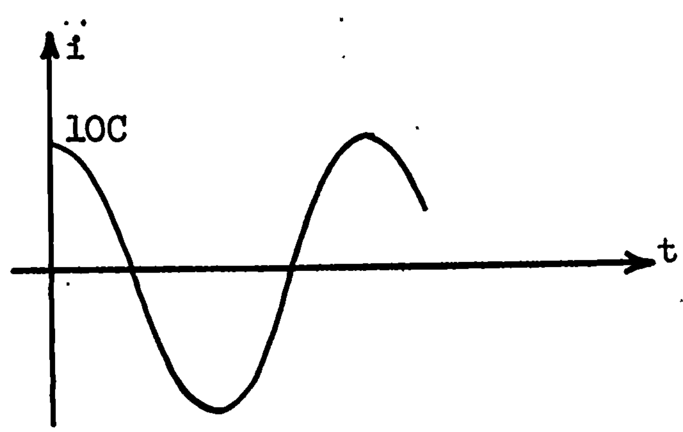
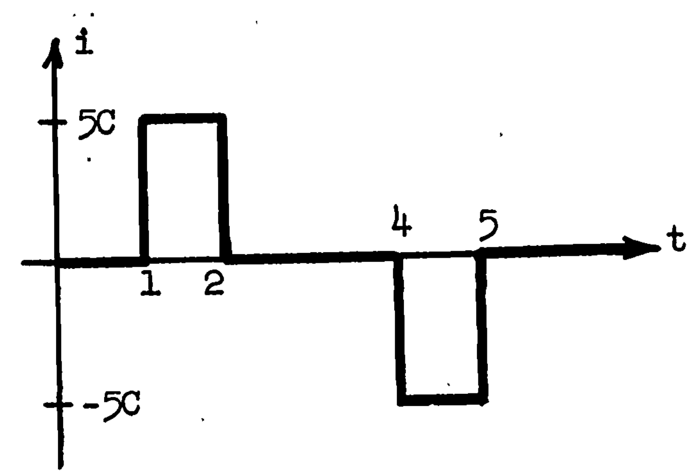
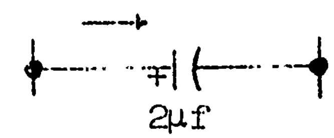
Answer:





Suppose the voltage v on a 2 μf capacitor is, successively, each of the functions given below. Write the expressions for the corresponding currents, with the following references:



(a) $v_1(t) = 100\ e^{-t/5}$ volts; $i_1(t) =$ ____________ ma.

(b) $v_2(t) = 45$ volts; $i_2(t) =$ ____________ ma.

(c) $v_3(t) = 10 \sin 1000t$ volts; $i_3(t) =$ ____________ ma.

<u>**Answer:**</u>

(a) $i_1(t) = -.04\ e^{-t/5}$ ma. $\left[2 \times 10^{-6}\left(-\frac{1}{5}\right) 100\ e^{-t/5} = -40 \times 10^{-6}\ e^{-t/5}\right.$ [illegible]$\left.\right]$

(b) $i_2(t) = 0$

(c) $i_3(t) = 20 \cos 1000t$ ma.

If the current through a capacitor is proportional to the derivative of the voltage across it, then, conversely, the voltage is ____________________
____.

18

<u>Answer</u>:

proportional to the <u>integral</u> (or antiderivative) of the current.

Specifically, the expression for the capacitor voltage in terms of its current is

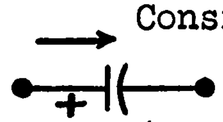
$v(t) =$ ______________________

20

<u>Answer</u>:

$$v(t) = \frac{1}{C} \int i(t)dt + K \text{ (a constant)}$$

(You may have left out the constant. If you did, go back and add it; it is essential.)

 Consider a 1 **μf** capacitance with the following voltage and current references: . Suppose there is initially a voltage of + 100 volts across the capacitor (having the same polarity as the reference). Then, a constant current of 0.1 ma is caused to flow in the reference direction. (You might think of an ideal constant current source being connected to the capacitor to make this happen.)

Find the value of the voltage after 0.2 second, assuming the current started at t = 0.

(a) v = ____________________.

Also, find the value of the voltage after 0.2 second, assuming the polarity of the <u>initial</u> voltage is reversed, everything else being the same.

(b) v = ____________________.

<u>Answer</u>:

(a) $v = 120$ volts

(b) $v = -80$ volts

This is obtained as follows:

$$v = \frac{1}{C} \int i \, dt + K = 10^6 \int 10^{-4} \, dt + K = 100t + K$$

To find K, note that when $t = 0$, $v = 100$; hence, $K = 100$ and so $v = 100t + 100$. Putting $t = .2$ here leads to the result.
If the polarity of the initial voltage is reversed, then when **evaluating** K we would say: when $t = 0$, $\underline{v = -100}$; hence, $K = -100$ and $v = 100t - 100$. Putting $t = .2$ leads to the result.

Finally, suppose the current in a .5 μf capacitor has the waveform given below, starting at time $t = 0$. Assuming the same references as before, find the expression for the corresponding voltage as a function of time.

$i(t) = \frac{1}{10} e^{-2t}$ ma. (Before the current starts, the capacitor voltage is 50 volts.)

$v(t) =$ ____________ volts.

Answer:

(a) **$v(t) = 150 - 100\ e^{-2t}$ volts.**

The solution goes like this:

$$v = \frac{1}{.5 \times 10^{-6}} \int \frac{1}{10} e^{-2t} (10^{-3}) dt$$

$$= 2 \times 10^{3} \left[- \frac{1}{20} e^{-2t} \right] + K = - 100\ e^{-2t} + K$$

at $t = 0$, $v = 50$ volts

$50 = -100 + K$; $K = 150$

Therefore, $v = 150 - 100\ e^{-2t}$ volts.

Another, slightly different form, used in expressing the capacitor voltage in terms of the current, can be found by integrating both sides of the expression

$$\frac{dv}{dt} = \frac{1}{C}\, i(t)$$

from some time t_1, considered to be a starting time, to an arbitrary time t. The resulting expression as an indicated integration will be:

____

Answer:

$$\int_{t_1}^{t} \frac{dv}{dt}\, dt = \frac{1}{C} \int_{t_1}^{t} i(t)dt$$

Remark:

Note that the upper limit of integration is t and so is the variable of integration -- what is usually called the "dummy" variable. It is a dummy because it disappears upon integration and substitution of limits. To illustrate,

$\int_a^b (8t+6t^2)dt = 4t^2+2t^3\Big|_a^b = 4b^2+2b^3-4a^2-2a^3$. In the final result the variable t, which was the dummy, does not appear. Thus, instead of t any other symbol can be used as a dummy variable. This is especially appealing if the same symbol appears in the limits of integration. In such cases, another symbol is sometimes used for the dummy variable to avoid confusion. Thus, the above result could be written

$\int_{t_1}^{t} \frac{dv}{dx}\, dx = \frac{1}{C} \cdot \int_{t_1}^{t} i(x)dx$ where x replaces t as the dummy variable. However, a different symbol does not carry the connotation of "time". Hence, at the risk of some confusion, we will continue to use t as the dummy variable even if t appears in the limits of integration.

The integration with respect to t on the left-hand side can be changed to an integration with respect to v by noting that $(dv/dt)\,dt = dv$. Since the integration is now in terms of v, the limits of integration must also be changed. The upper limit on the left will then be what v is at time t and the lower limit will be ______________________.

Write the resulting expression.

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Answer:

$$\int_{v(t_1)}^{v(t)} dv = \frac{1}{C} \int_{t_1}^{t} i(t)dt$$

The left-hand side can now be integrated and so the expression for the capacitor voltage in terms of the current becomes:

$v(t) =$ ______________________________.

30

<u>Answer</u>:

$$v(t) = \frac{1}{C} \int_{t_1}^{t} i(t)dt + v(t_1)$$

value of the capacitor voltage at the initial time t_1.

Note that $\int_{t_1}^{t} i(t)dt$ is simply the charge conducted to the capacitor by the current during the time interval from t_1 to t. Thus, the expression for the voltage can be interpreted in words as follows: The capacitor voltage at any time after the initiation of current through it equals the sum of two quantities: (1) the voltage resulting from the flow of charge over an interval of time plus (2) the value of the capacitor voltage at the time of initiation of the current.

The expression $\int_{t_1}^{t} i(t)dt$ is a definite integral in contrast to the indefinite integral previously obtained relating v and i. Write both expressions.

$v(t) =$ ____________________

$v(t) =$ ____________________

32

<u>Answer</u>: $v(t) = \frac{1}{C} \int i(t)dt + K$

$$v(t) = \frac{1}{C} \int_{t_1}^{t} i(t)dt + v(t_1)$$

(Examine these expressions carefully and note their similarities and differences.)

Very often the time at which the current is initiated is conveniently taken to be zero time. Write the appropriate expression for the capacitor voltage in the definite integral form for this case.

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Answer:

$$v(t) = \frac{1}{C} \int_0^t i(\tau)dt + v(0).$$

Suppose a $2 \mu f$ capacitor has an initial charge of 0.1 millicoulomb when at time $t = 0$ a current $i = 8\, e^{-t/10}$ microamps. is caused to flow. Find an expression for the capacitor voltage at any time thereafter.

$v(t) =$ ____________________

Also state the value of the capacitor voltage after the passage of a very long time.

Answer:

$$v(t) = 90 - 40\, e^{-t/10} \text{ volts}$$

$$v(t) \xrightarrow[t \to \infty]{} 90 \text{ volts}$$

(Verify by inserting $t = 0$ that the expression for $v(t)$ does reduce to the value of the capacitor voltage at the initial time.)

Solution

$$V_0 = \frac{q}{C} = \frac{.0001}{2\times10^{-6}} = 50 \text{ volts at } t=0$$

$$v = \frac{1}{C}\int i\, dt + K_0$$

$$= \frac{1}{2\times10^{-6}}\int 8\times10^{-6}\, e^{\frac{-t}{10}}\, dt + K$$

$$= -40\, e^{-t/10} + K$$

To summarize: a new device has been introduced called a ________________. It is characterized by a parameter C called the __________ and measured in _____________.

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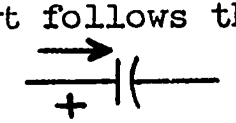
Answer:

capacitor

capacitance

farads (or microfarads)

The charge on one plate is proportional to the voltage across the capacitor. From this, it follows that the voltage-current relationship for the following references () is either

$i(t) =$ ____________________

or

$v(t) =$ ____________________.

40

Answer:

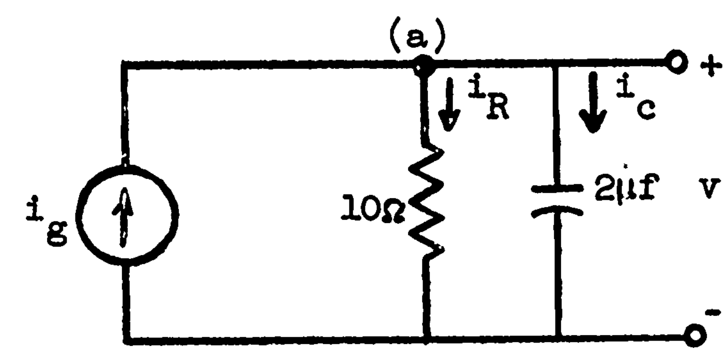
$$i(t) = C \frac{dv}{dt}$$

$$v(t) = \frac{1}{C} \int i \, dt + K$$

or

$$v(t) = \frac{1}{C} \int_{t_1}^{t} i(t) dt + v(0).$$

(Now, you are equipped to write the current and voltage relationships in simple circuits.)



$i_g = 10 \cos 200t$

Fig. 5

3. Simple RC Networks

You are now prepared to analyze the characteristics of networks which include capacitors, as well as sources and resistors. In such networks there is only one new factor which was not present when you analyzed resistive networks, namely the v-i relationship of the capacitor. All other principles still apply Indeed, Kirchhoff's two laws do not depend on what the branches are made up of, but only on the existence of branches forming closed paths and nodes. Hence, they will apply in networks containing capacitors, among other things.

Consider the network in Fig. 5. Write Kcl at the upper node (a) in terms of i_R and i_C.

i_g = ____________________

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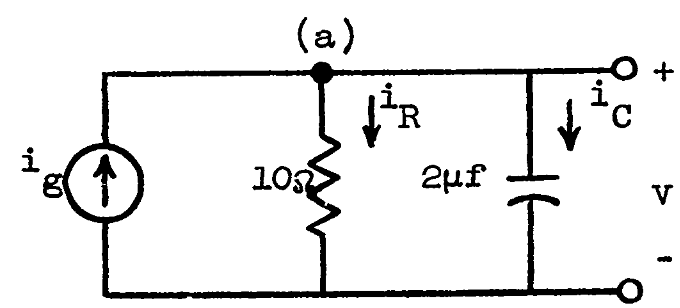
Answer:

$i_g = \underline{i_R + i_C}$



$i_g = 10 \cos 200t$

Fig. 5

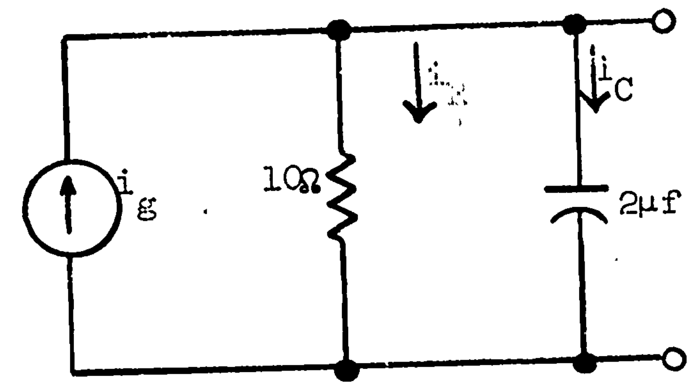
Suppose that at $t = 0$ we know the current i_C is to be 6 amps. Find the value of v at this time.

$v =$ ________________

Answer:

v = 40 volts. (At $t=0$ and $i_C(0)=6$ amps.)

(This is obtained by finding the value of i_R from the previous Kcl equation after putting $t = 0$ in $10 \cos 200t$, then using Ohm's law.)



$i_g = 10 \cos 200\ t$

Fig. 5 (Repeated)

$i_g = 10 \cos 200t$

Fig. 5 (repeated)

Next, give the rate at which the voltage v is changing with time. That is, write the value of dv/dt at this time.

$$\frac{dv}{dt} = \underline{\hspace{3cm}} \text{ at } t = 0 \text{ .}$$

<u>Answer</u>:

$\frac{dv}{dt}$ = <u>3 x 10^6 volts/sec.</u> at t = 0.

(This follows simply from the v-i relationship of the capacitor.)

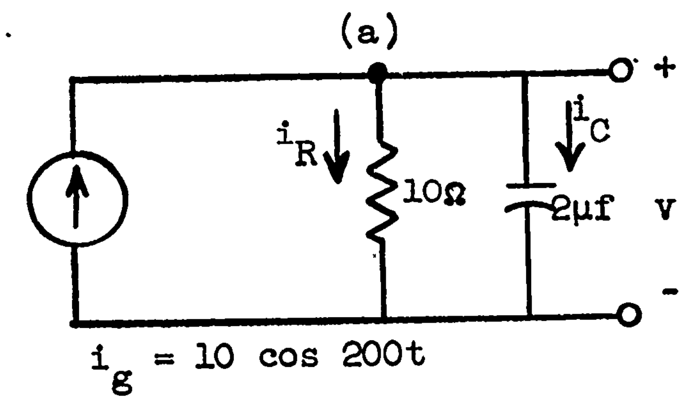


Fig. 5

The pattern that appears in the preceding development is similar to that which was used in resistive circuits in which the basic laws were applied alternately in computing voltages and currents. Here, also, the same pattern is followed with the added factor that in addition to <u>Kcl</u>, <u>Kvl</u> and <u>Ohm's law</u>, there is now available the v-i relationship of the capacitor as well: <u>$i = C\,dv/dt$</u>.

When dealing with resistive networks, the relationships involved are all <u>algebraic</u>. But since the v-i relationship of a capacitor involves a derivative, when carrying out the analogous procedure in networks containing capacitors, it may sometimes be necessary to differentiate in order to arrive at solutions.

As an illustration, let's again consider Fig. 5. This time suppose that di_C/dt is specified to be 10^3 amps/sec. at $t = 0$. <u>It is required to find the value of i_C at this time</u>. As before, applying Kcl gives: $i_g(t) = i_R(t) + i\ (t)$. (The functional notation is deliberately introduced to emphasize that this is valid at any time. But di_C/dt is the specified quantity. How can di_C/dt be obtained from this Kcl equation. State what to do and do it.

48

Answer:

Differentiate the Kcl equation.

$$\frac{di_g(t)}{dt} = \frac{di_R(t)}{dt} + \frac{di_C(t)}{dt}$$

Now, at $t = 0$, di_C/dt has been given numerically as 10^3 amps/sec. Furthermore, i_g is given as $10 \cos 200t$, and its derivative can be found. With this information $\frac{di_R}{dt}$ can be evaluated at $t = 0$. It will be:

$$\frac{di_R}{dt} = \_\_\_\_\_\_\_\_\_\_\_\_ \text{ at } t = 0.$$

50

Answer:

$$\frac{di_R}{dt} = -\underline{10^3 \text{ amps./sec.}} \text{ at } t = 0.$$

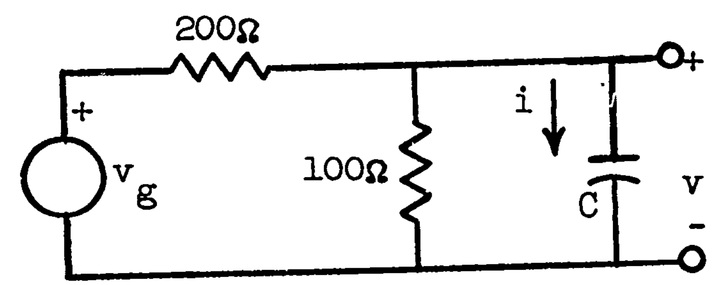
Remember that what we really want is the value of i_C at $t = 0$. What we have obtained is di_R/dt. But i_R is proportional to v by Ohm's law, and i_C is related to v. Using these facts, i_C at $t = 0$ is found to be

i_C = ________________ at $t = 0$.

Answer:

$i_C = -$ .02 amps. at $t = 0$.

$(i_C = C\ dv/dt = C\ d(Ri_R)/dt = RC\ di_R/dt)$



$v_g = 100 \cos 100\pi t/3$

Fig. 6

To illustrate further the alternate use of the basic relationships (Kcl, Kvl and the v-i relationships for resistors and capacitors), consider the network shown in Fig. 6. When $t = 10$ milliseconds, it is found that the capacitor voltage is $v = 30$ volts and its rate of change is $dv/dt = -10^7$ volts/sec. It is desired to find the value of C for which these values will hold.

Note that from the v-i relationship $i = C\, dv/dt$, C can be found if both i and dv/dt are known. The latter is given numerically. An expression for i, valid for any time, can be obtained in terms of other branch currents by ________________. When Ohm's law is subsequently used to express these currents in terms of appropriate branch voltages, this expression for i which holds at any t becomes:

$$i = \_\_\_\_\_\_\_\_ \ \_\_\_\_\_\_\_\_\_\_ .$$

Answer:

applying Kcl.

$$i = \frac{v_g - v}{200} - \frac{v}{100}$$

$$\left(\text{or } i = \frac{100 \cos 100\pi t/3 - v}{200} - \frac{v}{100}\right).$$

DATA

At t = 10 milliseconds,

v = 30 volts

$v_g = 100 \cos 100\pi t/3$

$\frac{dv}{dt} = -10^7$ volts/sec.

Inserting numerical values, this becomes $i =$ ____________ amps. (You should get $i = -0.2$ amps.) Finally, the value of C becomes:

$C =$ _______________ μf.

Answer:

$$C = 0.02\ \mu f.$$

Solution:

$$i = \frac{100\cos\pi/3 - 30}{200} - \frac{30}{100}$$

$$= .10 - .30 = - .20 \text{ amps.}$$

$$i = C\frac{dv}{dt} = - .20 = C(-10^{7})$$

$$C = .02 \times 10^{-6} f = .02\ \mu f$$

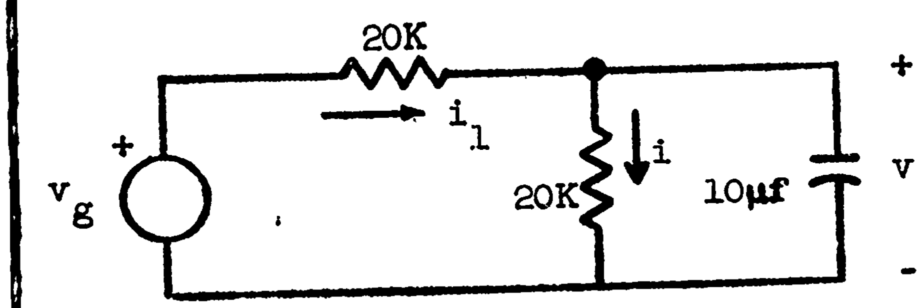


Fig. 7

2

Up to this point emphasis has been on numerical values at a given point in time. However, the same principles apply to general solutions as functions of time. Thus, in Fig. 7 suppose that current i is specified to be

$$i(t) = \frac{1}{100}\,(1 - e^{-t/10})$$

and it is desired to find v_g as a function of time.

As a first step v_g can be obtained in terms of the voltages across the two resistors by Kvl, and these, in turn, can be expressed in terms of the currents by Ohm's law. The result is

______________________________.

58.

Answer:

$$v_g = 20{,}000\,(i_1 + i)$$

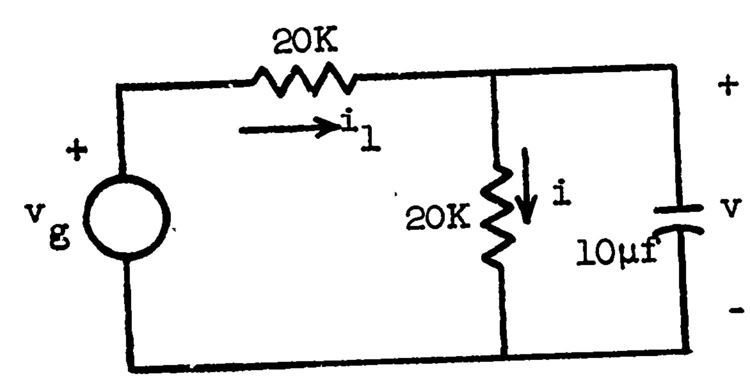


Fig. 7

But i_1 can be expressed in terms of i and the capacitor current, by Kcl. The capacitor current, in turn, can be expressed in terms of the voltage v by means of the v-i relationship of the capacitor. Finally, v is related to i by Ohm's law. As a result of all these, the previous expression for v_g can be written:

$$v_g = \underline{\hspace{6cm}}$$

(This should be in terms of i only.)

60

Answer:

$$v_g = \underline{40{,}000 \left(i + \frac{1}{10}\frac{di}{dt}\right)} \text{ volts.}$$

Reminder

$$i(t) = \frac{1}{100}\left(1 - e^{-t/10}\right)$$

Finally, inserting the specified value of $i(t)$ leads to

$$v_g = \underline{\hspace{6cm}}.$$

62

Answer:

$$v_g = 400\,(1 - .99\,e^{-t/10})$$

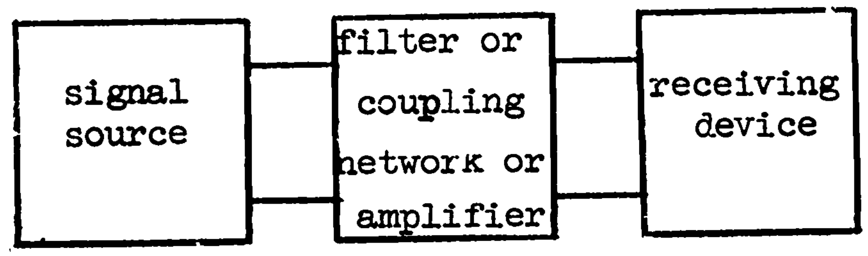


Fig. 8(a)

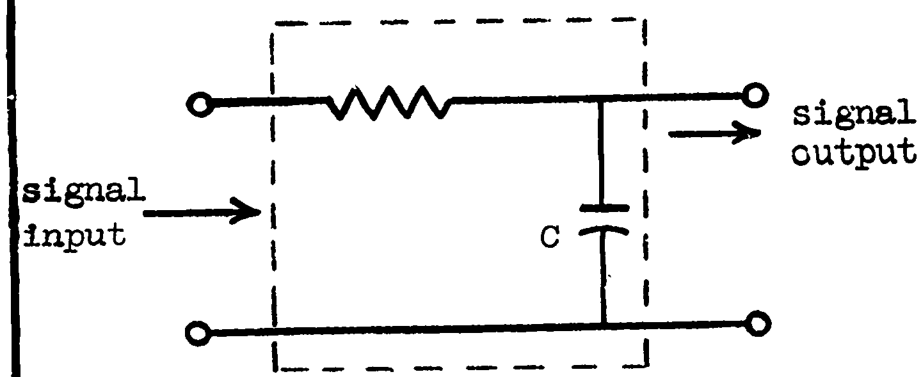


Fig. 8(b)

Up to this point, in considering networks in which capacitors appear, it has been assumed that a voltage or current in a capacitor or resistor is known either at some instant of time or as a function of time. However, the more usual problem is to determine how the voltages and currents in such a network vary with time after some initial instant at which a source is switched on or off. In such a case some additional factors enter into the problem. We shall next devote attention to this class of problems.

The situation illustrated in Fig. 8 is representative of a large number of applications. A signal which starts at some time is to be transmitted through some kind of transmitting network to a receiving device of some kind. It is necessary to know how the signal (which is simply a current or a voltage) is modified by the transmitting system.

A particularly simple case is shown in Fig. 8(b). In the following pages, we shall analyze the characteristics of this circuit in considerable detail. In order to consider a number of possible variations all at the same time, we shall deal with the situation illustrated in Fig. 9.

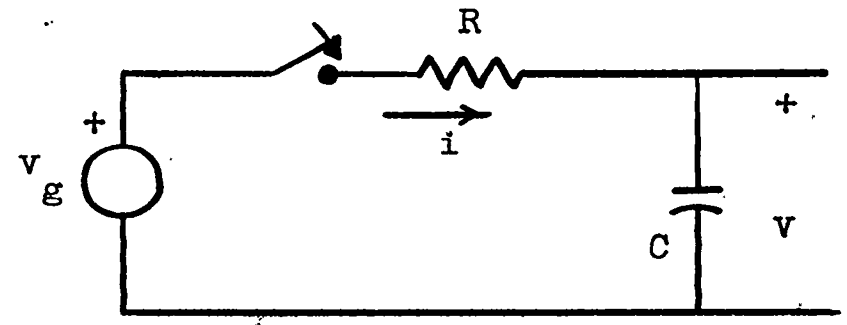


Fig. 9

Assume that the capacitor C, in Fig. 9, has been charged at some time in the past to a voltage V_o. At a particular time, which can be designated $t = 0$, the switch is closed, thereby applying the voltage source v_g, which is the source of the signal, to the R and C network. (It is assumed that v_g is known.) It is desired to find expressions for the capacitor voltage v and the current i as a function of time following the closing of the switch.

As a first step, write Kvl around the loop, at the same time eliminating the resistor voltage by Ohm's law.

v_g = ____________________

66

Answer:

$$v_g = Ri + v$$

(where the capacitor v is some function of the initial voltage V_0 and the source voltage v_g.)

In this expression both v and i appear and both are unknowns. Either one of them can be eliminated by using a relationship (Kvl, Kcl, Ohm's law and the v-i relationship of the capacitor) that has not yet been used in arriving at this equation.

State the relationship which should now be used.

68

Answer:

The v-i relationship of the capacitor.

In writing an equation for this v-i relationship, we have a choice. In the expression $v_g = Ri + v$, i and v are the capacitor current and voltage.

We can insert an appropriate expression either for i or for v, and thus obtain two different equations. One of these expressions would lead to a solution more easily than the other. Write out both of these equations.

a) $v_g =$ ______________________

b) $v_g =$ ______________________

Answer:

i

a) $v_g = \underline{R\,(C\,\frac{dv}{dt})} + v$

v

b) $v_g = \underline{Ri + (\frac{1}{C}\int i(t)dt + K)}$

or

$v_g = \underline{Ri + \frac{1}{C}\int_0^t i(x)dx + v(0)}$

Let us temporarily concentrate on the first of these:

$$RC \frac{dv}{dt} + v = v_g$$

Here we find something new; this is not a simple <u>algebraic</u> equation. Not only does the unknown, v, appear in the equation, its <u>derivative</u> appears also. An equation like this in which one or more derivatives of an unknown appear is called a <u>differential equation</u>.

In the list below check those equations which are <u>differential equations</u>, just to be sure that the above definition of a differential equation is clear.

a) $2y + 5\frac{dy}{dt} = 0$

b) $4y + 3x + \frac{d}{dx}(5 \sin x) = 0$

c) $2v + \int e^{-2t}dt = 6i$

d) $a\frac{d^2i}{dt^2} + bv = ci$

<u>Answer</u>:

a) and d) are differential equations.

b) is not a differential equation, even though an indicated differentiation appears, because there is no derivative of an unknown, i.e., performing the indicated operations would <u>not</u> lead to an expression containing derivatives.

Note that in one of the equations in the last frame, there was a <u>first derivative</u> and in the other there was a <u>second derivative</u>. In general, derivatives of <u>any order</u> may appear.

We define the <u>order</u> of the differential equation as the order of the highest derivative appearing in the equation. Thus, if the highest derivative that appears is the third derivative, the differential equation is of third order. The following differential equation:

$$5\frac{d^2v}{dt^2} + 3\frac{dv}{dt} + 2v = 0$$

is of ____________ order.

74

<u>Answer:</u>

<u>second</u>

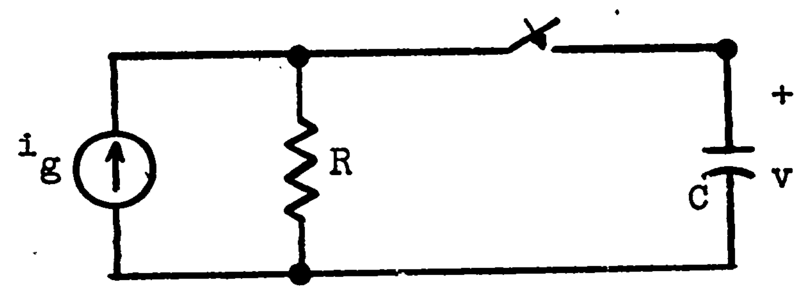


Fig. 10

We are going to be concerned with such differential equations which arise when the basic electrical laws are applied to RC networks. To gain some experience in writing the equations from a given network, let's consider a few more cases.

In Fig. 10 is shown a network in which a switch is closed at a particular time taken to be $t = 0$. It is desired to find a single equation containing only the capacitor voltage as a variable, like the one found for the previous network. Redraw the network after the switch is closed. State three different procedures by which the desired equation can be obtained.

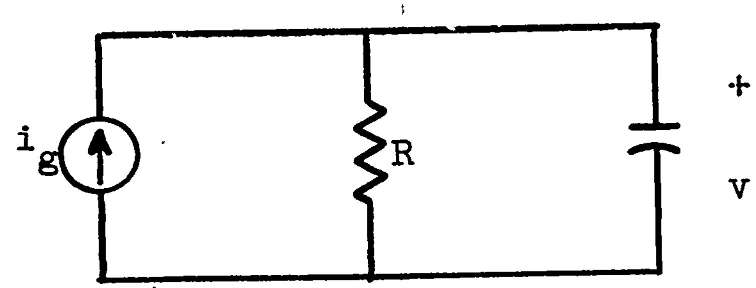
There are several ways that you could attack this problem, to obtain an expression for the capacitor voltage. You have used these different methods many times. State three procedures by which the desired equation can be obtained.

1.

2.

3.

Answer:



1. Convert the current source and R into a voltage source equivalent, thus arriving at the same structure as the previous example.
2. Write a Kvl equation around the loop formed by R and C.
3. Write a node equation.

Let's consider the last two possibilities in some more detail, taking first the loop equation.

Let i be the capacitor current with such a reference that $i = +C\,dv/dt$.

1) Write the loop equation and use Ohm's law to replace the resistor voltage. (Watch out for the resistor current!)

2) Use an appropriate relationship to substitute for i in terms of v.

<u>Answer</u>:

1) $R(i - i_g) + v = 0$

or

$Ri + v = Ri_g$

2) $RC \frac{dv}{dt} + v = Ri_g$

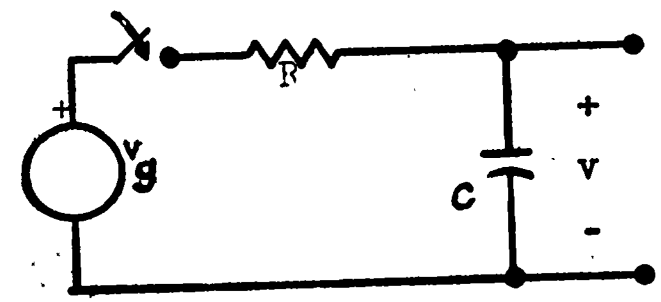


Fig. 9 (Repeated)

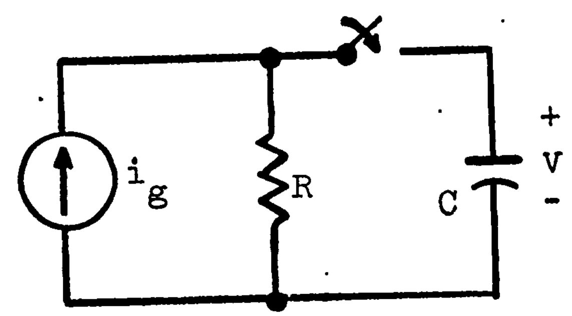


Fig. 10

This expression is much like the one obtained for the previous network of Fig. 9

$$RC \frac{dv}{dt} + v = v_g$$

The only difference is the appearance of Ri_g instead of v_g.

Now let's write a node equation for Fig. 10. (When writing the algebraic sum of the currents leaving a node, express the branch currents in terms of the node voltages.)

The node equation is

______________________________.

80

<u>Answer:</u>

$$C\frac{dv}{dt} + \frac{v}{R} = i_g$$

or

$$RC\frac{dv}{dt} + v = Ri_g$$

(You should feel good that you got the same answer by using two different approaches to the problem.)

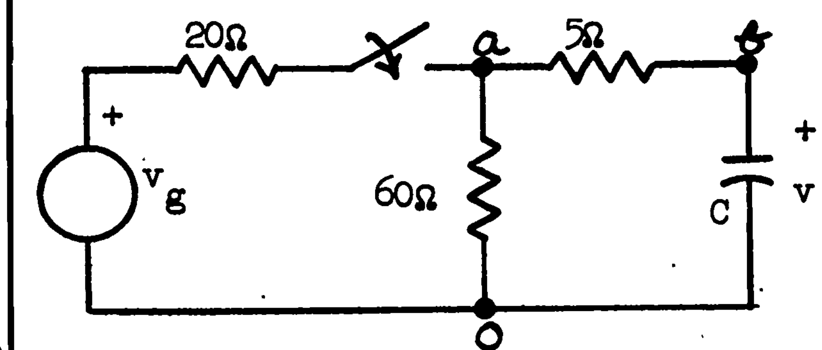


Fig. 11

As a final example, let's consider the network in Fig. 11. (Numerical values are used for the resistances to avoid undue algebraic complications.) At $t = 0$, the switch is closed and it is desired to find an equation for the voltage across the capacitor. One possible method of procedure is to replace everything but the capacitor by a Thevenin equivalent and thereby arrive at a series R and C, just like the first network we considered. (Do this as a check on what you will do in the following.)

Two other procedures are:

a) ______________________________

b) ______________________________

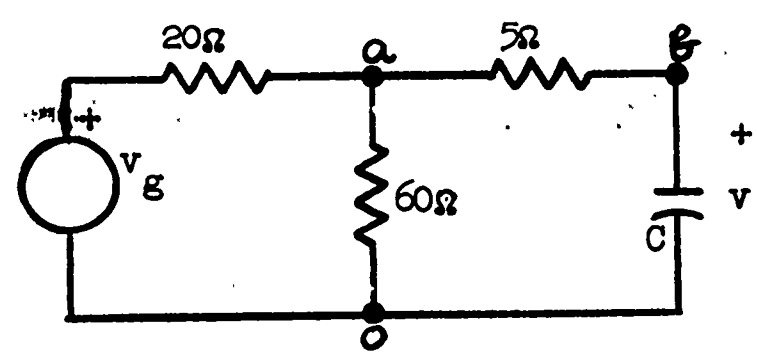


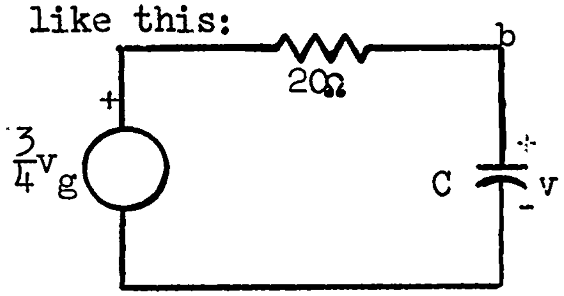
Fig. 11
(after switch is closed)

<u>Answer</u>:

a) Write a set of loop equations.

b) Write a set of node equations.

(The Thevinin equivalent circuit would look like this:



$$20\,C\,\frac{dv}{dt} + v = \frac{3}{4}\,v_g$$

If node 0 is chosen as a datum and node equations are written at nodes a (node-to-datum voltage = v_a) and b (node-to-datum voltage = v), the result will be:

$$\left.\begin{aligned} &\text{node a:} \quad \frac{v_a - v_g}{20} + \frac{v_a}{60} + \frac{v_a - v}{5} = 0 \\ &\text{node b:} \quad \frac{v - v_a}{5} + C\,\frac{dv}{dt} = 0 \end{aligned}\right\} \quad \text{or} \quad \left\{\begin{aligned} &v_a - \frac{3}{4}\,v = \frac{3}{16}\,v_g \\ &v - v_a + 5\,C\,\frac{dv}{dt} = 0 \end{aligned}\right.$$

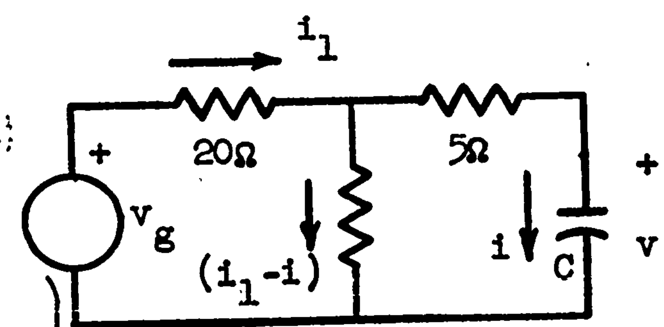
This is a set of two equations in two unknowns, v_a and v. Since we are interested in v we eliminate v_a by solving for v_a from the second equation and substituting into the first. The result of these steps is:

$$\left(5\,C\,\frac{dv}{dt} + v\right) - \frac{3}{4}\,v = \frac{3}{16}\,v_g \qquad \text{or} \qquad 20\,C\,\frac{dv}{dt} + v = \frac{3}{4}\,v_g$$

Note the similarity between this equation and the corresponding ones for the previous networks.

The final possibility is to write a set of loop equations and to carry out a procedure similar to the one just completed. (You'll have to eliminate the capacitor current appropriately. Carry out this procedure; let's see if you can arrive at the same final result.

Answer:



1. First choose loop currents (see diagram). Then write Kvl equations around the two "windows".

$$\left\{\begin{array}{l} 20i_1 + 60(i_1-i) = v_g \\ \\ -60(i_1-i) + 5i + v = 0 \end{array}\right\} \text{or} \left\{\begin{array}{l} 80i_1 - 60i = v_g \\ \\ -60i_1 + 65i + v = 0 \end{array}\right\}$$

2. Then solve for i_1 from, say, the second equation, and substitute into the first:

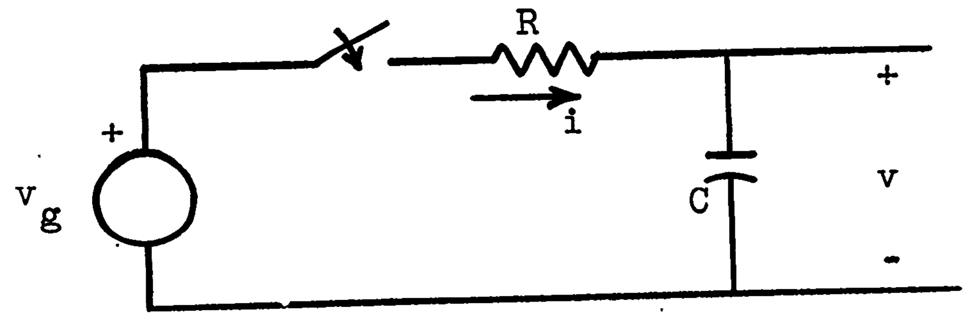
$$80\left(\frac{65}{60} i + \frac{v}{60}\right) - 60\ i = v_g \qquad \text{or} \qquad 20i + v = \frac{3}{4} v_g$$

3. Finally, replace i by C dv/dt and get

$$20\ C \frac{dv}{dt} + v = \frac{3}{4} v_g$$

(The last two steps can be interchanged.)

What you might suspect from these examples is that in a **network** having a single capacitor and source, but any number of resistors, the equation for the capacitor voltage is a differential equation of <u>first order</u>. And your suspicions would be right. Thus, by a detailed study of the network of Fig. 9, we will at the same time be covering a large number of other cases as well.



Kvl; $Ri + v = v_g$

or $RC\dfrac{dv}{dt} + v = v_g$

Fig. 9

Let's now return to the equation for the capacitor voltage in Fig. 9.

$$RC\frac{dv}{dt} + v = v_g$$

In order to carry out a solution, it will be necessary to know specifically what the source voltage v_g is.

Let us first assume that the source is a battery so that v_g is a constant: $v_g = V_1$, so that the equation becomes

$$RC\frac{dv}{dt} + v = V_1$$

In order to solve for v, this equation should somehow be integrated.

Two students discussed this problem, each proposing a different approach:

Plan 1: Write the equation as $\frac{dv}{dt} = -\frac{1}{RC}(v - V_1)$ and integrate both sides with respect to t.

Plan 2: Write the equation as in Plan 1, then divide both sides by $v - V_1$, and then integrate with respect to t. That is, write

$$\frac{1}{v-V_1}\frac{dv}{dt} = -\frac{1}{RC}$$

then integrate.

Which plan would you choose?

Plan 1: go to page 88.

Plan 2: go to page 89.

This one won't work because v is really v(t) which is an unknown function of time. If we try to integrate $-(v - V_1)/RC$, we can write

$$\int -\frac{(v - V_1)}{RC}\,dt = -\frac{1}{RC}\int v\,dt + \frac{V_1}{RC}\int dt$$

The second term can be integrated; but the integrand of the first term is v, and v is not known as a function of time; so it can't be integrated. Go back and choose the other plan.

This plan will work. But as we prepare to integrate, we must first decide whether to find an antiderivative (indefinite integral) or to find the definite integral from the initial time $\tau = 0$ to a later time t. Remembering that

$$\frac{1}{v-V_1}\,\frac{dv}{dt} = -\frac{1}{RC}$$

let's do both in order to compare their results. In each case carry out the integration on the right and show the indicated integration on the left.

1. Indefinite integral ______________________________

2. Definite integral ________________________________

90

<u>Answer</u>:

Indefinite integral:

$$\int \frac{1}{v-V_1} \frac{dv}{dt} \, dt + K = -\frac{t}{RC}$$

(The constant of integration <u>must</u> appear. If you did not have it, return and insert it.)

Definite integral:

$$\int_0^t \frac{1}{v-V_1} \frac{dv}{dt} \, dt = -\frac{t}{RC}$$

Again note that $(dv/dt)dt = dv$, and so the integration with respect to t can be changed to one with respect to v. In the case of the definite integral, the limits of integration must also be changed appropriately. Do this and carry out the integration for both cases. The result will be:

1. Indefinite integral ______________________

2. Definite integral ______________________

<u>Answer</u>:

1. Indefinite integral: $\underline{\ln [v(t) - V_1] + K = -\frac{t}{RC}}$

2. Definite integral: $\underline{\ln \frac{v(t) - V_1}{V_0 - V_1} = -\frac{t}{RC}}$

(Note, the integrands become $dv/(v-V_1)$ which is of the form du/u, whose antiderivative is $\ln u$. In the case of the definite integral, the limits are V_0 (lower) and v (upper). The functional notation, $v(t)$, is used for emphasis.)

The first of these can be reduced to the second by evaluating the constant K. Thus, in the first expression let $t = 0$. The value of $v(t)$ at this time is __________, and so K becomes

K = ____________________

<u>Answer</u>:

$v(t)$ at $t = 0$ is $\underline{v(0) = V_0}$.

$$K = - \ln (V_0 - V_1)$$

Now substitute this value of K back into the original expression

$$\ln (v(t) - V_1) + K = - t/RC$$

and, remembering that $\ln x - \ln y = \ln x/y$, convince yourself that both integrals lead to the <u>same</u> result; namely,

$$\ln \frac{v(t) - V_1}{V_0 - V_1} = - \frac{t}{RC}$$

In this form, v is not given explicitly in terms of t. As a final step take the antilogarithm of both sides and solve for $v(t)$.

$v(t) =$ __________________ for $t \geq 0$

<u>Answer</u>:

$$v(t) = \underline{V_1 + (V_0 - V_1)\, e^{-t/RC}} \qquad \text{for } t \geq 0$$

or

$$v(t) = \underline{V_0\, e^{-t/RC} + V_1(1 - e^{-t/RC})}$$

Note: V_1 is the applied battery voltage and V_0 is the initial capacitor voltage.

This expression gives the capacitor voltage explicitly as a function of time. It remains to examine the nature of this relationship or the <u>shape</u> of the plot of the output voltage as time passes after the closing of the switch.

First, consider the exponential $e^{-t/RC}$. Since the exponent of an exponential should be dimensionless, what is the dimension of the product RC?

98

Answer:

RC has the dimension of <u>time.</u>

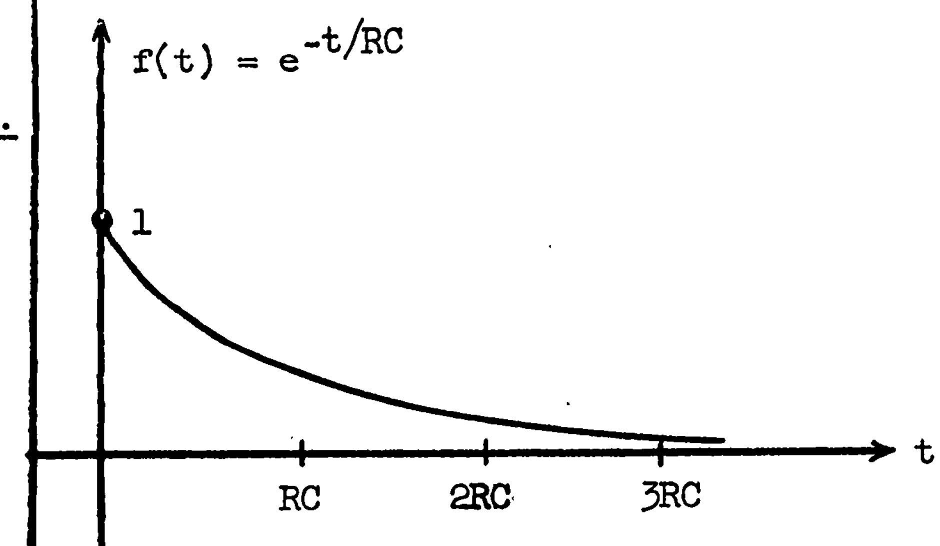


Fig. 12

For this reason RC is called the time constant. Figure 12 is a plot of the exponential $f(t) = e^{-t/RC}$. Evaluate the slope of this function at $t = 0$.

Slope of $e^{-t/RC}$ at $t = 0$ is ______________________.

Draw a straight line having this slope tangent to the curve at $t = 0$, and extend it until it intersects the t-axis. From the geometry, state the value of t at which this line intersects the axis.

Answer:

$$\text{slope} = -\frac{1}{RC}$$

The straight line intersects the axis at $t = RC$.

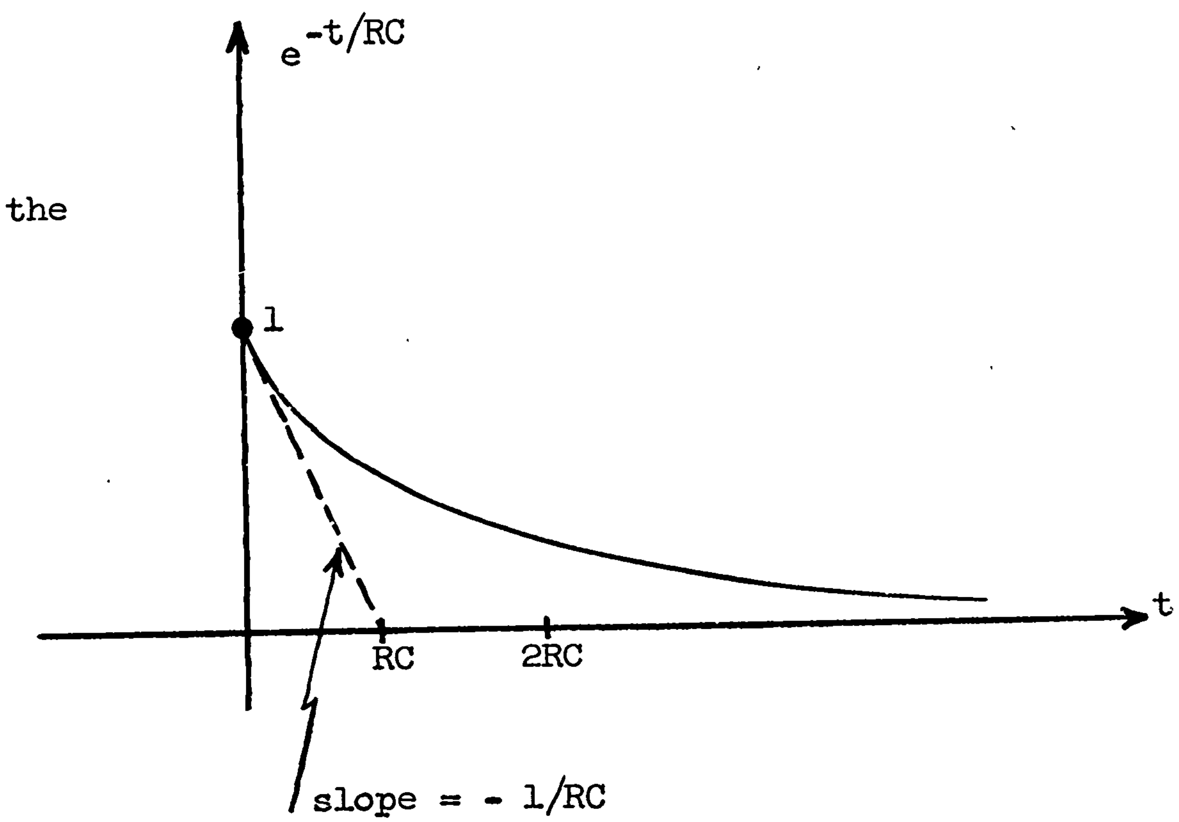


Fig. 12(a)

Thus, the slope of the line tangent to the exponential at $t = 0$, is the negative reciprocal of the ______________________.

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<u>Answer</u>:

time constant, RC (Say it out loud. Remember it. We'll be using it many times.)

We shall use T (<u>capital</u> tee) as a general symbol for the time constant. The time constant can be taken as a unit of time. Thus, the time axis in the plot of the exponential was labeled in terms of units of RC. Suppose the time constant is 4 milliseconds; then an interval of 10 milliseconds can be said to last for 2 and 1/2 time constants.

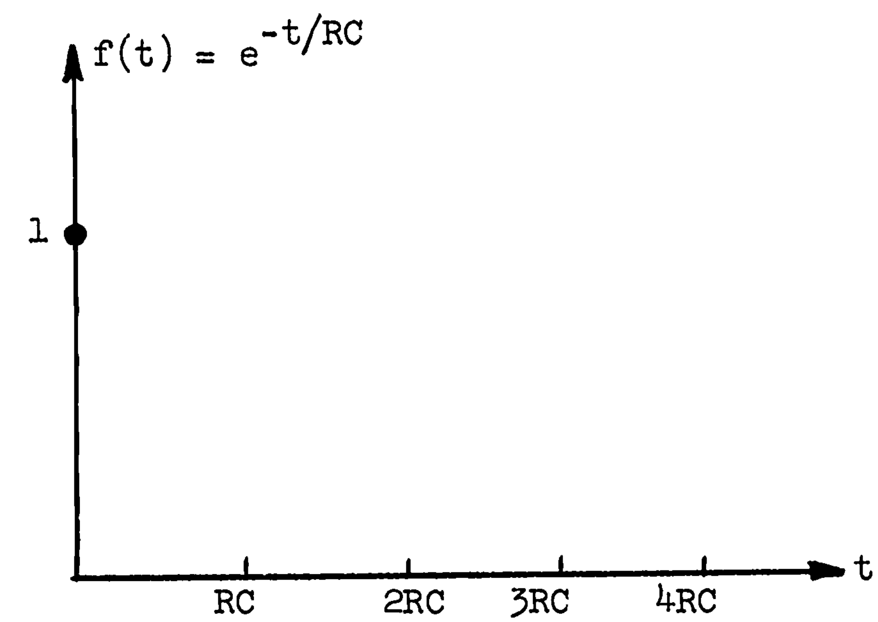
In Fig. 12(a) how many time constants elapse for the <u>line drawn tangent to the exponential curve at t = 0</u> to drop to zero?

_______________ time const.

104

Answer:

<u>One</u> time const.



Let's examine the nature of this exponential function.

When $t = T$, that is when $t = RC$, then $v(t) = e^{-1}$. Since $e^{-1} = 1/e = 0.37$, the exponential curve has dropped from 1.0 to 0.37 in the elapsed time of one time constant (when $t = T = RC$). This drop amounts to 63 per cent of the total drop which will occur as time continues to infinity. As $t \rightarrow \infty$, the exponential function approaches zero.

What percentage of the total change from $t = 0$ to ∞ is completed by the exponential in (a) 3 time constants and (b) 4 time constants.

a) ________________ per cent

b) ________________ per cent

Draw the approximate shape of this curve on the axes provided on page 104.

<u>Answer</u>: (a) <u>95</u> per cent

(b) <u>98</u> per cent

(Note that $e^{-3} \doteq 0.05$ and $e^{-4} \doteq .02$)

<u>Remark</u>

For many practical purposes 2 per cent can often be considered negligible. In fact, in making most electrical measurements, the normal accuracy of our measurements is often of the order of 1 or 2 per cent. Hence, it can often be assumed that an exponential has completed its change in <u>4 time constants</u>. This is a useful number to remember.

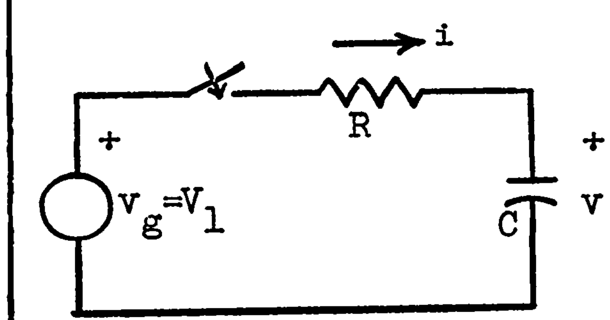


Fig. 9

Now let's return to the expression for the capacitor voltage in Fig. 9 which we found on page 96:

$$v(t) = V_0 e^{-t/T} + V_1(1 - e^{-t/T}) \qquad \text{for } t \geq 0$$

where $T = RC$ is the time constant. Here the result is grouped in two terms. The first term has as a multiplier the initial voltage V_0 of the capacitor; the second has a multiplier which is __________________. (Write it in words.)

It is possible to view the whole solution in two parts using the superposition principle: first, consider that the capacitor has a nonzero initial voltage V_0 but that the source voltage is zero (de-activated); then, consider that __________

____.

108

Answer:

the source voltage V_1

the source voltage has a nonzero value V_1 but that the initial capacitor voltage is zero.

Reminder

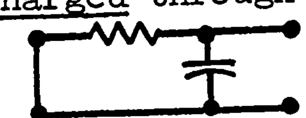
The expression is:

$$v(t) = V_0 e^{-t/T} + V_1(1 - e^{-t/T})$$

Write the separate expressions for the capacitor voltage for each of these two cases:

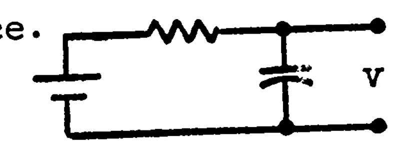
1. Initially charged capacitor being <u>discharged</u> through a resistor:

   $v(t) =$ ______________ for $t \geq 0$

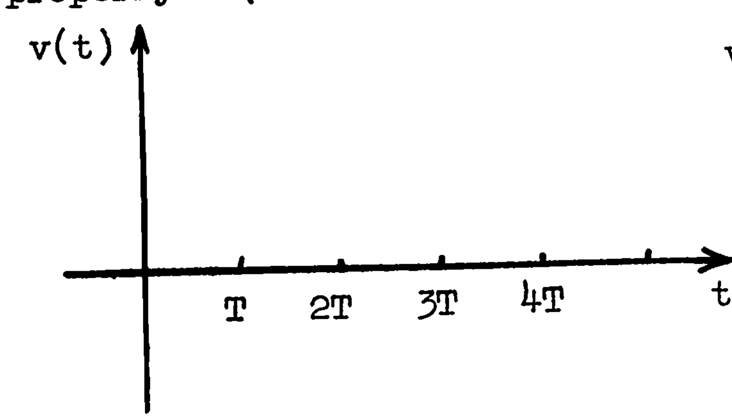


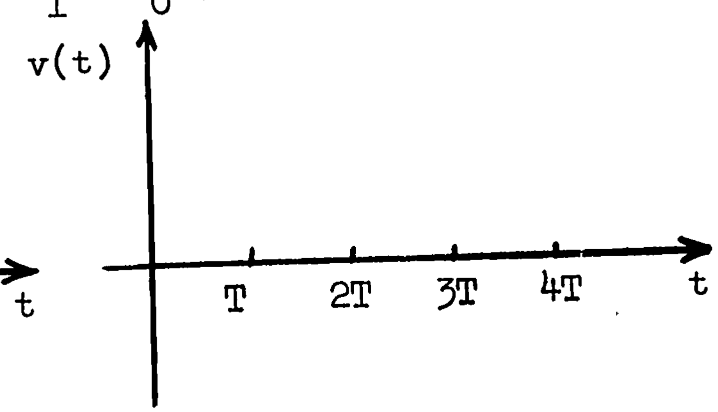
2. Initially uncharged capacitor being charged through a resistor by a suddenly applied constant voltage source.

   $v(t) =$ ______________ for $t \geq 0$



Also sketch waveforms of $v(t)$ for these two cases. Label the pertinent points properly. (For definiteness, let $V_1 > V_0$.)



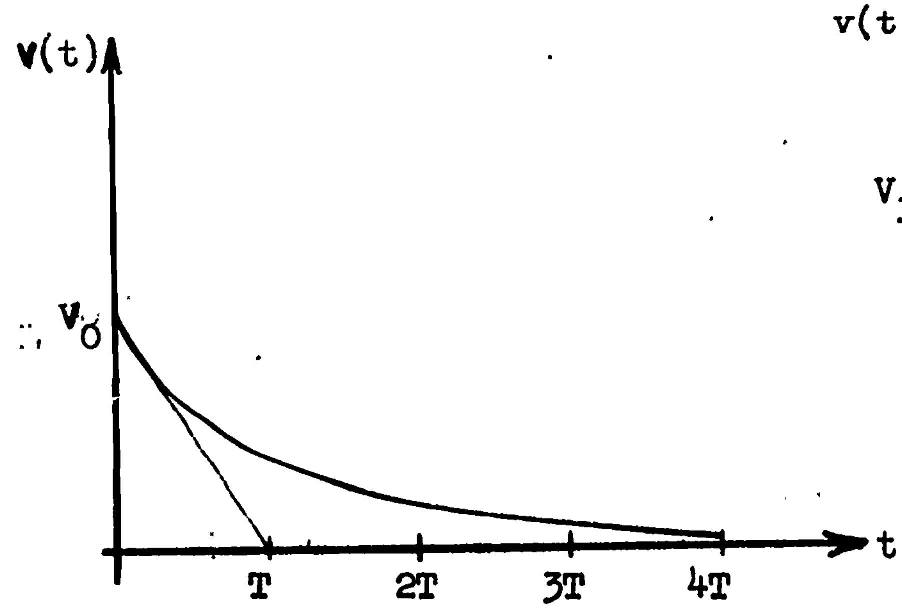


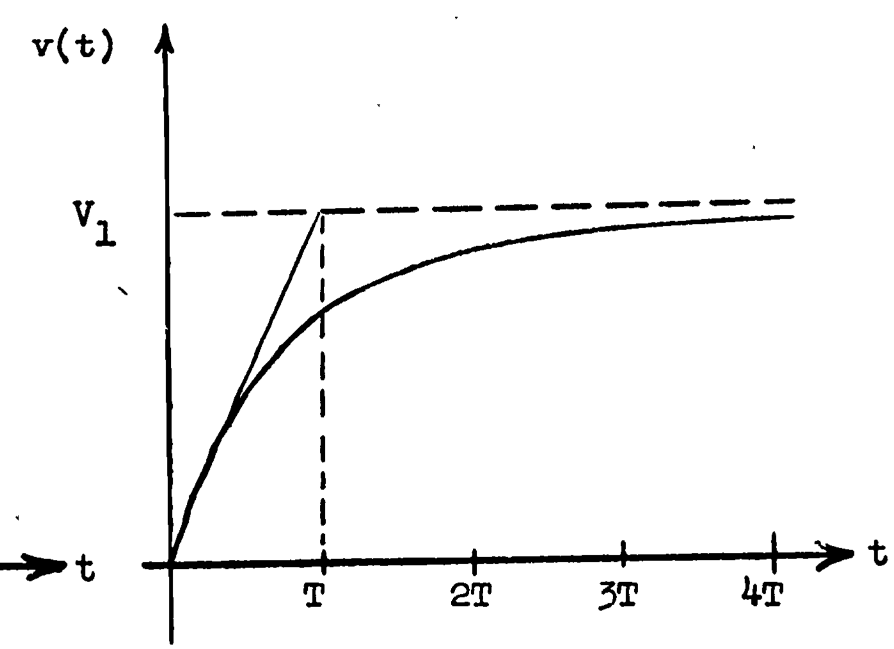
110

Answer:

1. $v(t) = V_0 \, e^{-t/T}$

2. $v(t) = V_1(1 - e^{-t/T})$





In the case of the discharging capacitor, the curve clearly shows that the capacitor voltage starts at its initial value V_0 and drops exponentially to 2 per cent of its initial value in __________ (how many) time constants. In its final state, the capacitor voltage reaches _____________.

On the other hand, the voltage of the charging capacitor starts from zero and rises ___________ (in what manner?) toward its final value, which equals ____________.

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<u>Answer</u>:

<u>4</u> time constants

<u>zero</u>

<u>exponentially</u>

$\underline{V_1}$.

Let's consider the initially charged capacitor, which is discharging through a resistor, in more detail. The expression

$$v(t) = V_0 \, e^{-t/RC} = V_0 \, e^{-t/T} \qquad \text{for } t \geq 0$$

is an exponential. There are <u>two</u> constants in this expression which determine the curve of voltage against time:

1. the quantity V_0 (which is the ______________________________), determines the point on the voltage axis at which the curve starts at $t = 0$, and
2. the ________________, which determines the rate at which the curve falls to its final value, zero.

114

Answer:

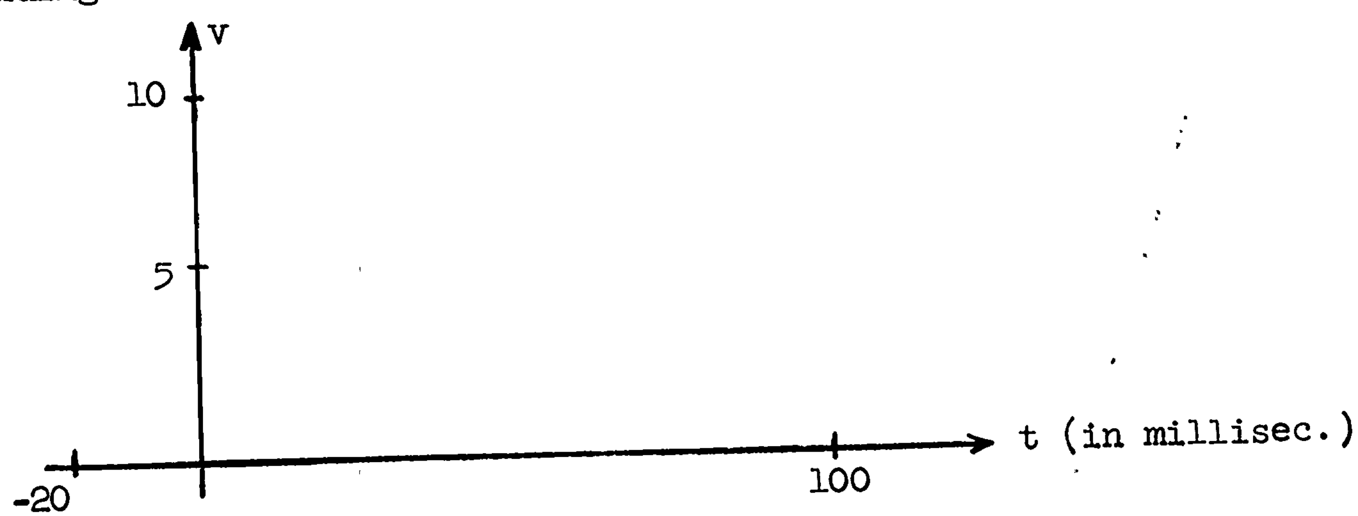
<u>the initial value of the capacitor voltage</u>

<u>time constant T</u>

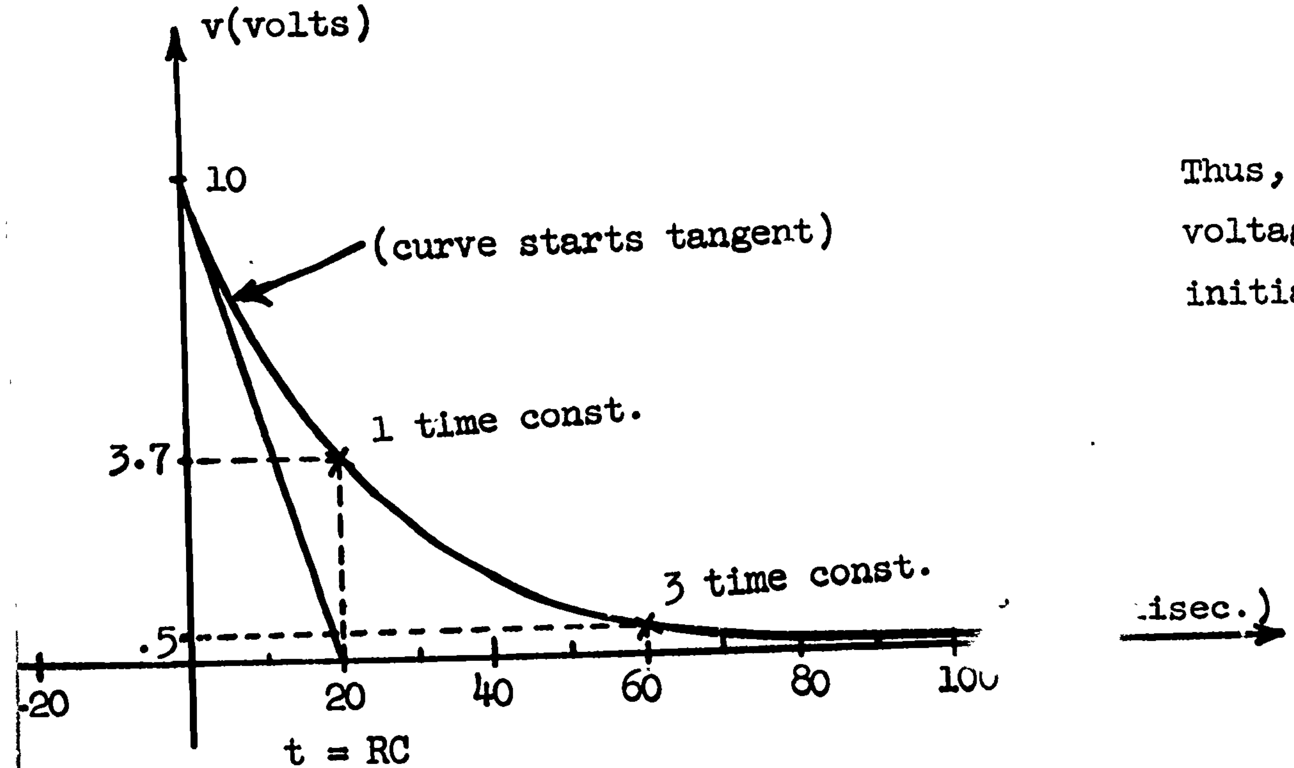
same final result.

Once these two quantities are known, together with the fact that the final value of the capacitor voltage is zero, the curve of the variation of v against t can be sketched.

Suppose a 2 µf capacitor is charged to an initial voltage of 10 v... for a long time. At t = 0 a switch is closed connecting a 10 kilohm resistor across the capacitor. Sketch the plot of the capacitor voltage approximately <u>to scale</u> from t = - 20 to + 100 milliseconds. Lable appropriate points, in particular the points corresponding to 1 and 3 time constants.



Answer:



Thus, to make a sketch of the capacitor voltage it is only necessary to know the initial voltage, V_0, the final voltage, and the time constant.

Let's turn next to the initially uncharged capacitor which is charged from a d.c. source through a resistor. The expression for the voltage following the switching is

$$v(t) = V_1\,(1 - e^{-t/T})$$

Here also there are just two constants which will determine the shape of the curve of the function: V_1, the source voltage and T, the time constant. In the present case, the initial value of the capacitor voltage, and its <u>final</u> (ultimate) value after the passage of a long time are:

a) initial value of v = __________

b) final value of v = __________

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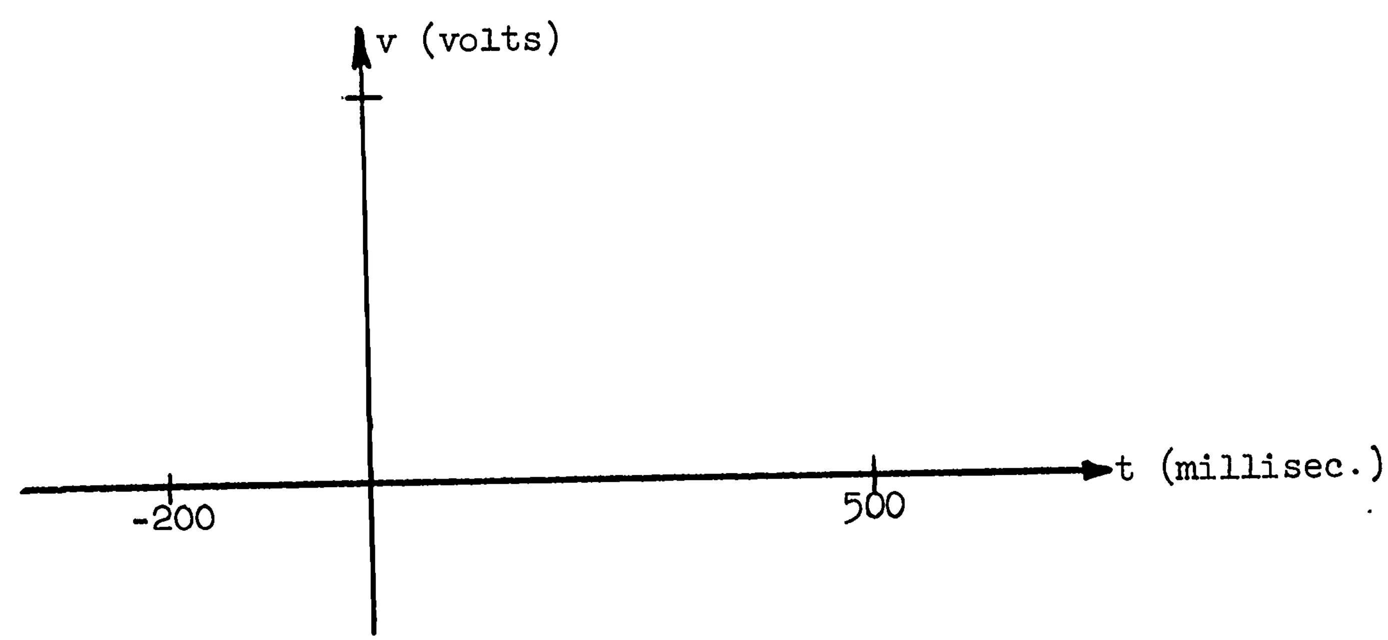
Answer:

initial value of v = $\underline{0}$

final value of v = $\underline{V_1}$

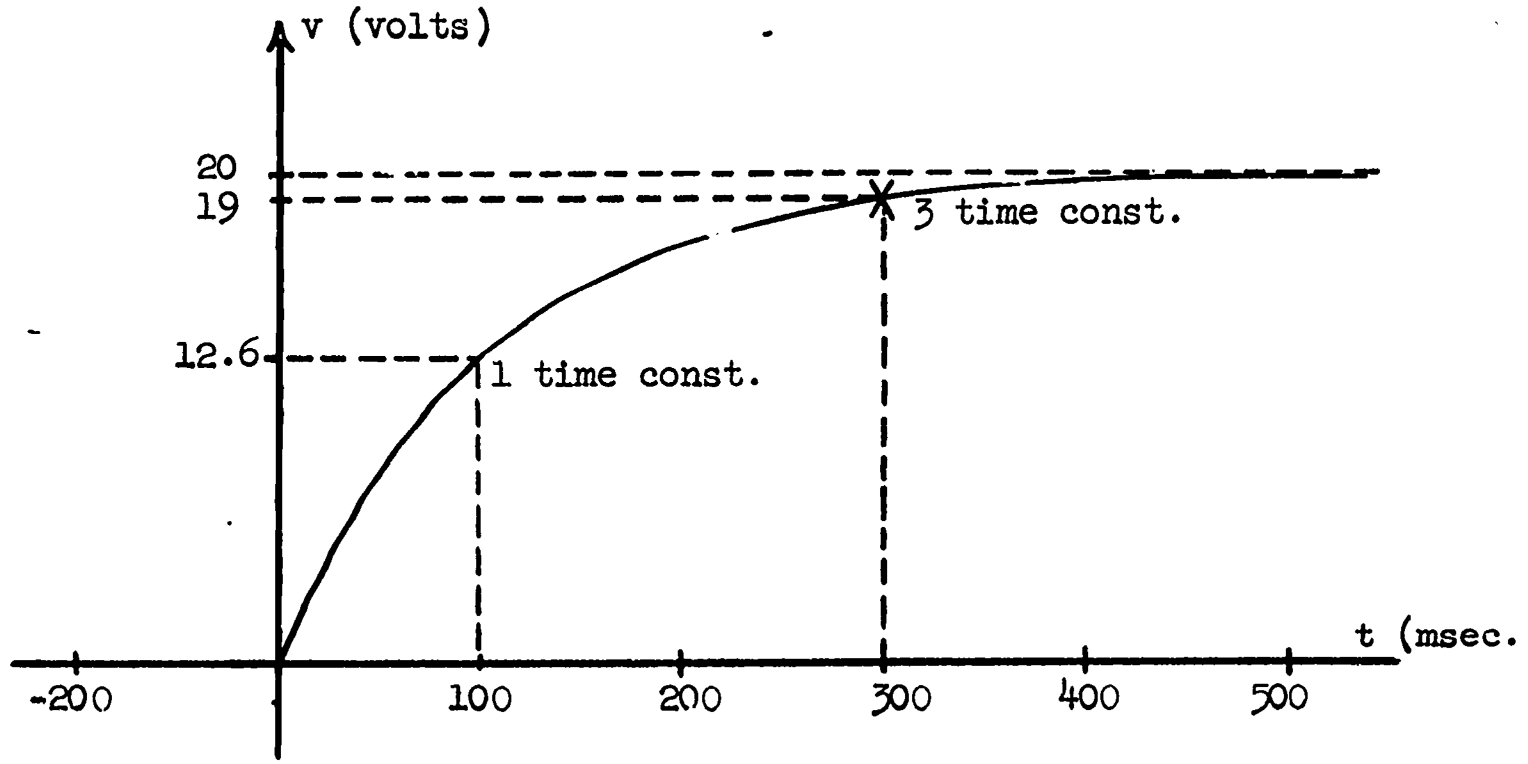
Knowing the initial value and the final value, a sketch of the voltage can be made as a function of time once the time constant is known, since the time constant determines the rate at which the curve goes from its initial to its final value.

Thus, suppose an initially uncharged .5 μf capacitor is charged from a 20-volt battery through a 200 kilohm resistor. Sketch the plot of the capacitor voltage approximately to scale for t = - 200 to + 500 milliseconds. Label appropriate points.



120

<u>**Answer:**</u>



Finally, let us return to the general situation in Fig. 9 in which there is <u>both</u> an initial voltage and a charging source. The capacitor voltage after the closing of the switch was earlier found to be

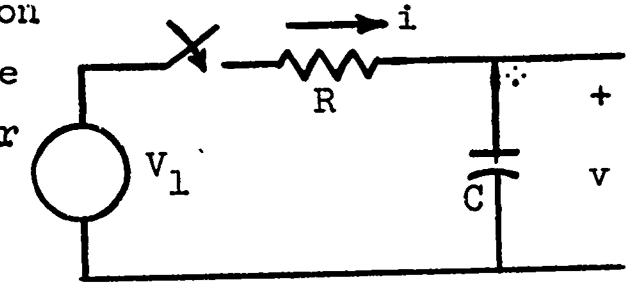


Fig. 9

$$v(t) = V_0\, e^{-t/T} + V_1(1 - e^{-t/T})$$

or

$$v(t) = V_0 + (V_1 - V_0)\,(1 - e^{-t/T})$$

(You should verify that these are different **forms** of the same expression by removing the parentheses.)

In this case the initial value and final value of the capacitor voltage are:

a) initial value = ________

b) final value = ________

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<u>Answer</u>:

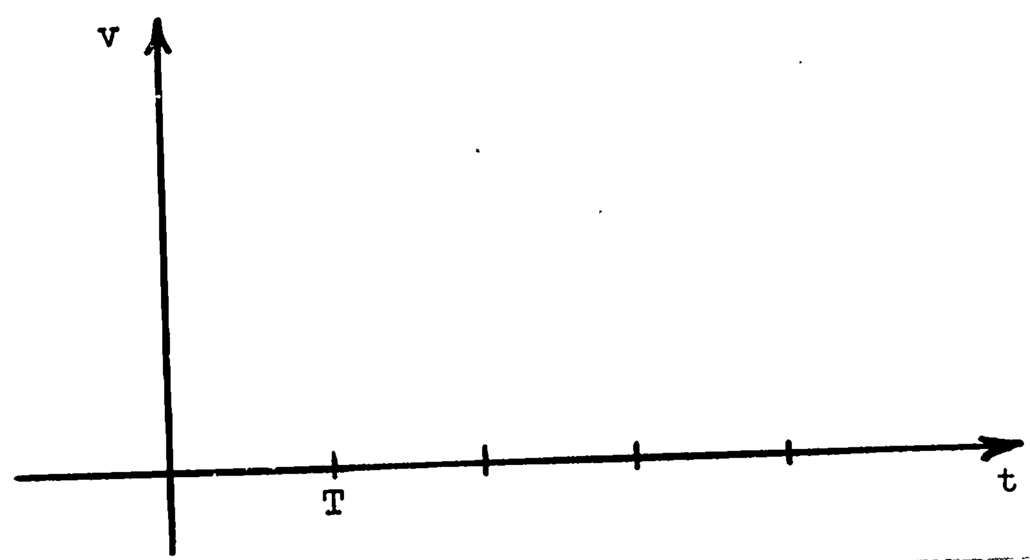
a) initial value = $\underline{V_0}$

b) final value = $\underline{V_1}$

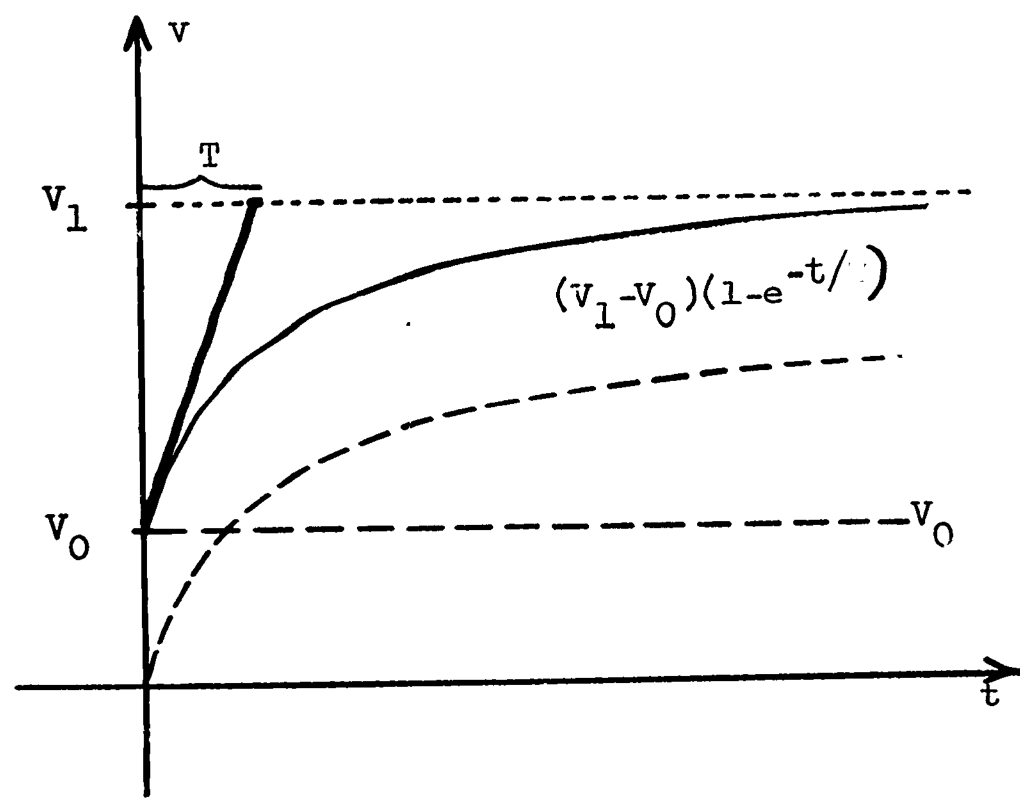
The expression

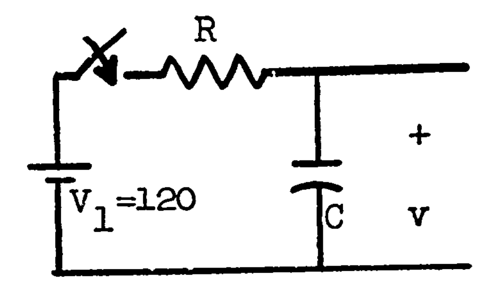
$$v(t) = V_0 + (V_1 - V_0)(1 - e^{-t/T})$$

is perhaps the simplest which can be used to sketch the voltage as a function of time. In this form, there are two additive terms. The first is a constant V_0; the second is like the case we have just completed. Sketch each of the two terms separately as dotted lines on a set of axes; then sketch their sum as a full line. Assume that $V_1 = 2.5\ V_0$.



<u>Answer</u>:





RC = 100µsec.

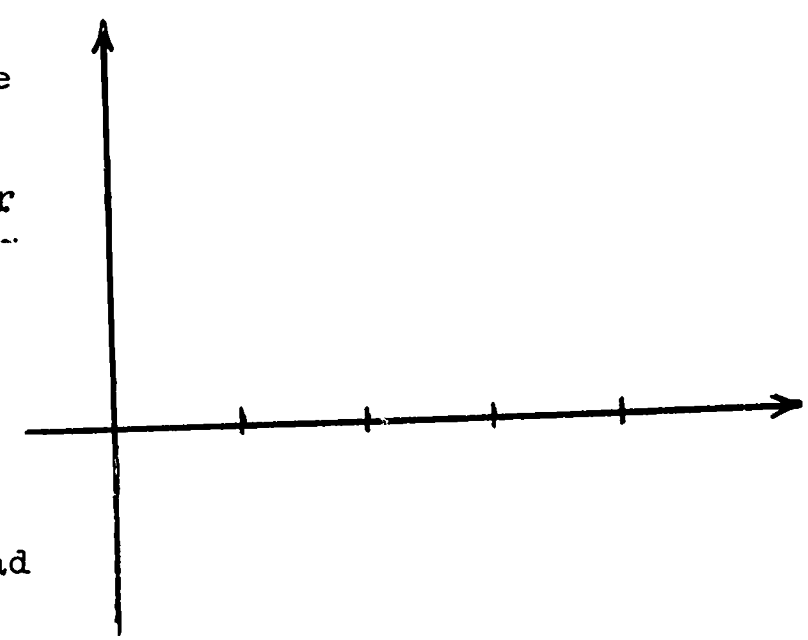
Here again we find that the voltage rises from its initial value to its final value exponentially. The rate at which it rises will depend on the time constant. If a sketch of the voltage is required, it is only necessary to know the initial and final values and the time constant; it is not necessary to solve a differential equation.

In Fig. 13, the capacitor has an initial voltage of + 50 volts when, at time $t = 0$, the switch is closed.

a) Write an expression for the capacitor voltage as a function of time following the closing of the switch.

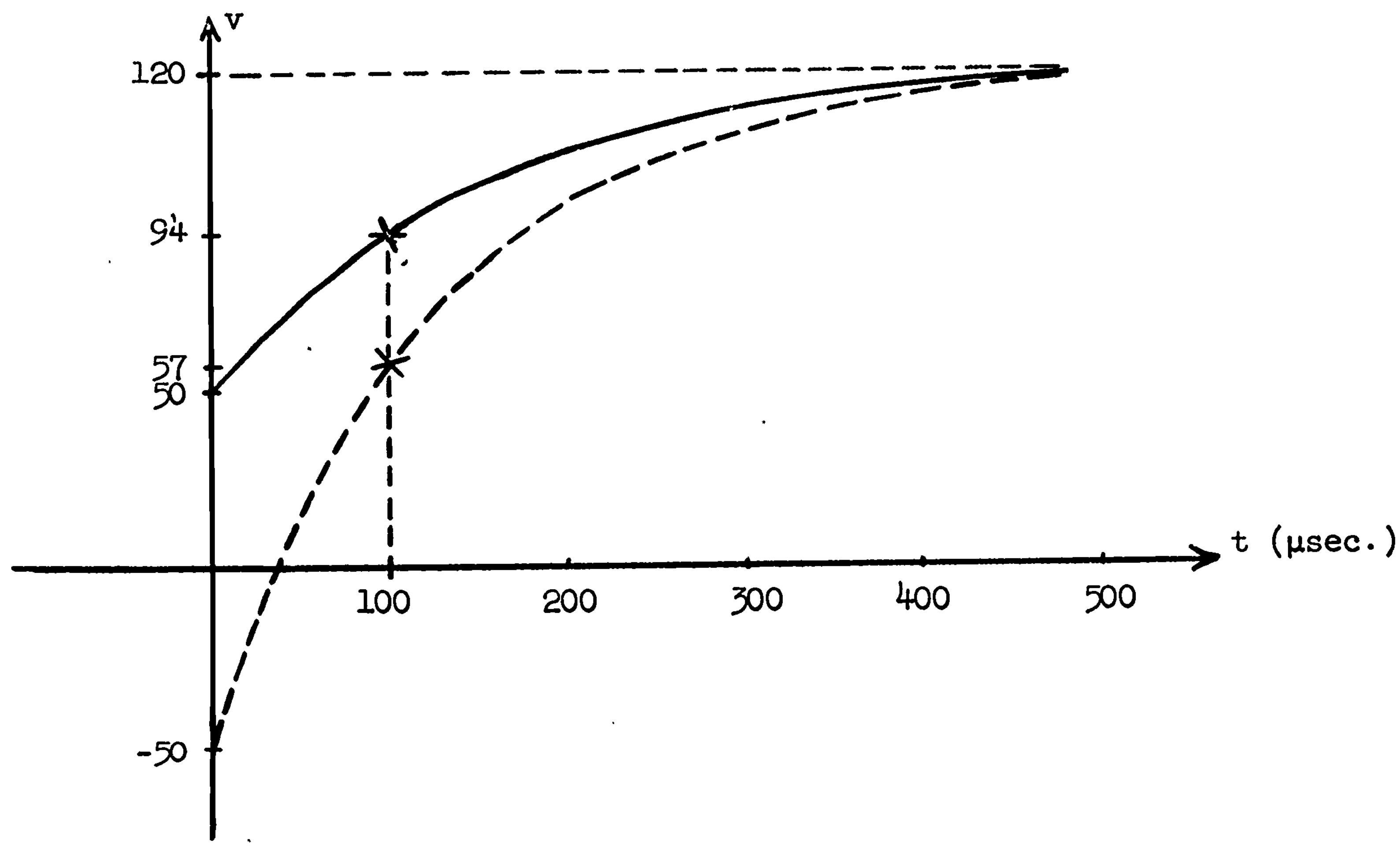
b) Sketch the waveform of the capacitor voltage, labeling all appropriate points.

c) On the same axes sketch in a dotted line the voltage waveform if the initial voltage is - 50 volts instead of + 50 volts.



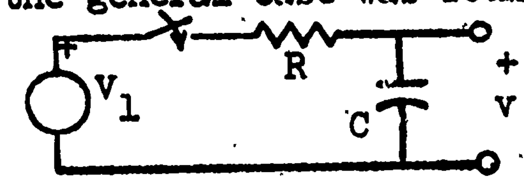
Answer:

$$v(t) = 50 + 70(1 - e^{-10^4 t})$$



When the problem of Fig. 9 was first started, it was the intention to examine both the capacitor voltage and its current. Having concentrated on the voltage, let us now turn to the current. The voltage in the general case was found to be

$$v(t) = V_0 e^{-t/T} + V_1(1 - e^{-t/T})$$



(Fig. 9)

Using the voltage-current relationship of the capacitor, and remembering what the time constant is in terms of R and C, write an expression for the current i(t).

i(t) = ______________________

<u>Answer</u>:

$$i(t) = \frac{V_1 - V_0}{R} e^{-t/T}$$

(Manipulate your answer until it agrees with this.)

For the two distinct cases: (a) $V_1 = 0$, $V_0 \neq 0$ and (b) $V_0 = 0$, $V_1 \neq 0$, the form of the expression is just the same, a <u>decaying</u> exponential.

Case (a) $\quad i(t) = - V_0 e^{-t/T} \quad V_1 = 0$

Case (b) $\quad i(t) = V_1 e^{-t/T} \quad V_0 = 0$

This is in contrast with the voltage, which is a <u>decaying</u> exponential in case ______ but a ________________ in the other case.

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<u>Answer:</u>

case <u>(a)</u>

<u>rising (or increasing, or growing) exponential</u>

Note that the <u>actual</u> direction of current in case (a) will depend on the polarity of the initial voltage. If V_0 is positive, meaning the initial voltage actually has the same polarity as the reference, then the current will be negative, meaning the actual current will be opposite to the reference.

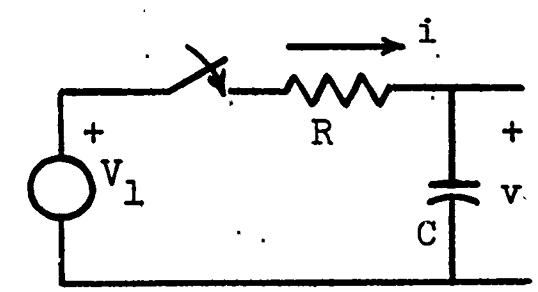


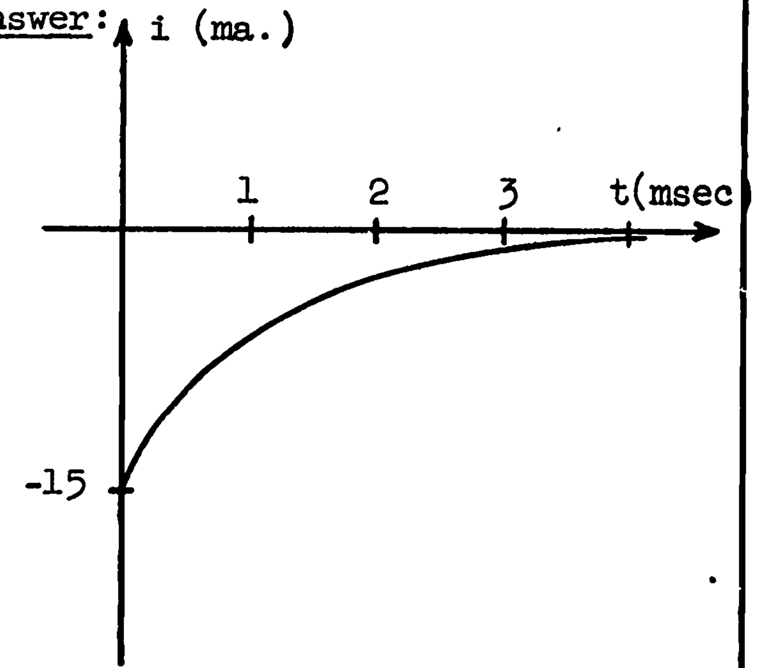
Fig. 9

In the general case when neither V_0 nor V_1 is zero, the expression for the current is still an exponential. Its sign will depend on the relative values of V_1 and V_0. Let's consider the circuit in Fig. 9 again.

Suppose the initial capacitor voltage is 25 volts and the dc source voltage is 10 volts, with R = 1 kilohm and C = 1μf. Sketch the curve of the current against time following the switching. Label the time axis in milliseconds and the current in milliamps. What is the actual direction of current relative to the reference direction?

132

<u>Answer</u>:



The <u>actual</u> current is opposite to the reference.

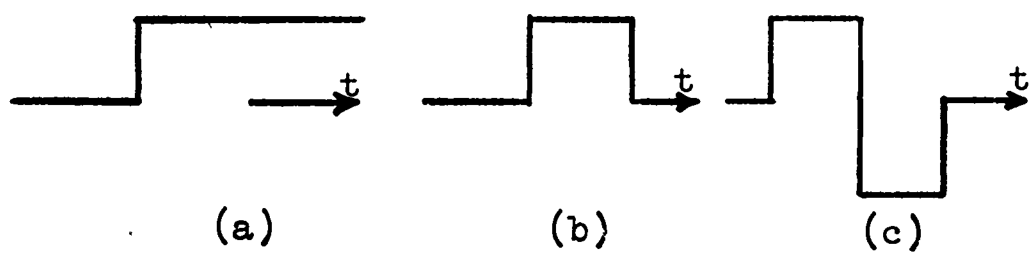


Fig. 14

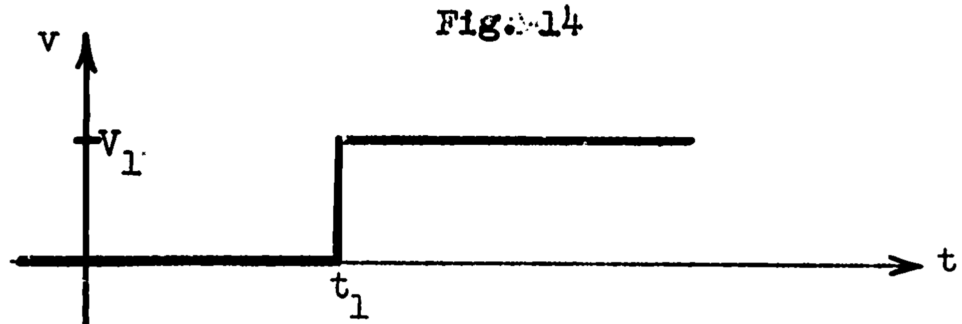


Fig. 15

Another item of unfinished business is the consideration of the case when the source voltage in Fig. 9 is something other than a constant. Some signal waveforms that are of great importance in communications and in computers are shown in Fig. 14.

The first one is called a <u>step function</u>. Up until some time, the voltage (assuming the signal is a voltage) is zero when it suddenly jumps to a non-zero value and stays there. This description fits the case of a battery suddenly switched into a network and indeed we can say in the case of Fig. 9 that a voltage <u>step function</u> is <u>applied</u> to the RC network.

Let's consider the step function in somewhat more detail. In order to describe the step function shown in Fig. 15, we would need to state two things: (a) <u>when</u> the discontinuous step takes place and (b) what the <u>magnitude</u> of the step is. In Fig. 15 these two quantities are:

a) the time at which the step occurs = ________________

b) the magnitude of the step = __________

134

Answer:

a) $\underline{t_1}$

b) $\underline{V_1}$

When the magnitude of a step is <u>unity</u>, it is called a <u>unit step function</u>. A unit step function occurring at time $t = t_1$ is denoted by $\underline{\mu(t-t_1)}$. Thus, mathematically, the unit step is expressed as

$$\mu(t-t_1) = 1 \qquad \text{for } t > t_1$$
$$= 0 \qquad \text{for } t < t_1$$

Draw a graph of the functions,

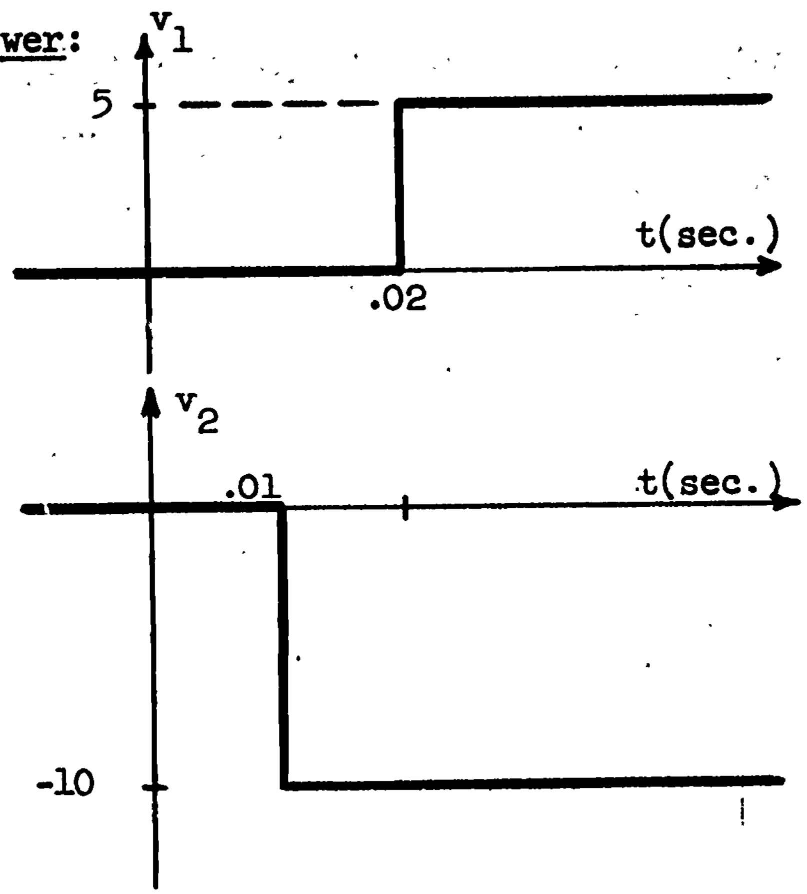
$$v_1 = 5u(t-.02)$$

and $$v_2 = -10u(t-.01)$$

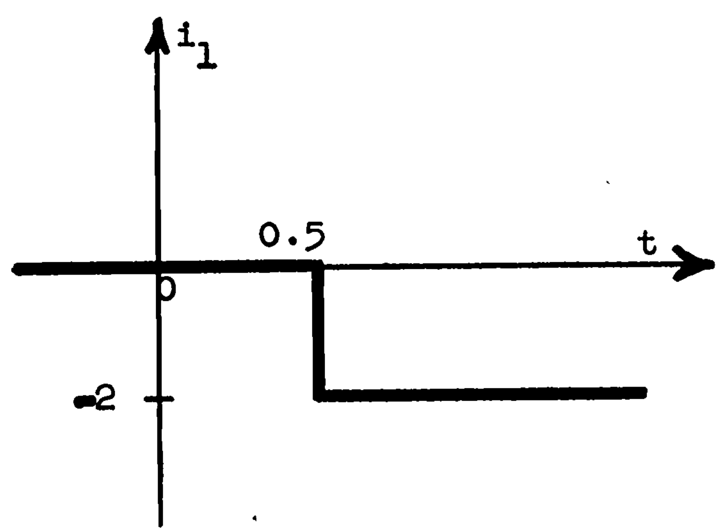
Label the points at which the steps occur.

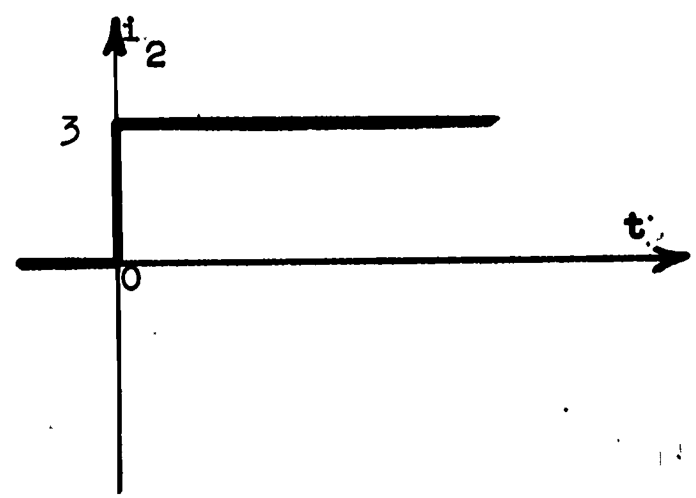
136

Answer:



Now write the mathematical expressions for the step functions shown graphically. We will take these to be steps of current rather than voltage so that you won't get the idea that there are only voltage steps.





$i_1(t)$ = ______________ $i_2(t)$ = ______________

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Answer:

$i_1(t) = -2u(t-.5)$

$i_2(t) = 3u(t-0) = 3u(t)$

(Note that the unit step function occurring at $t = 0$ is simply $u(t)$.)

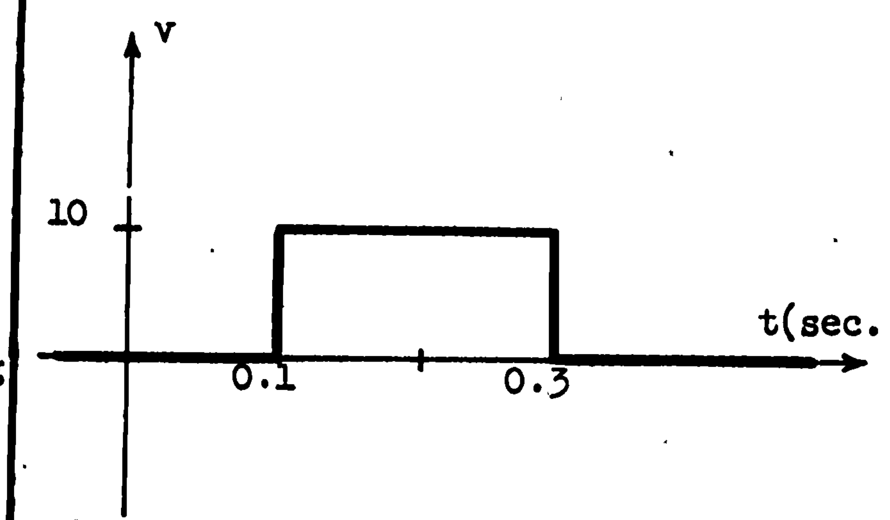
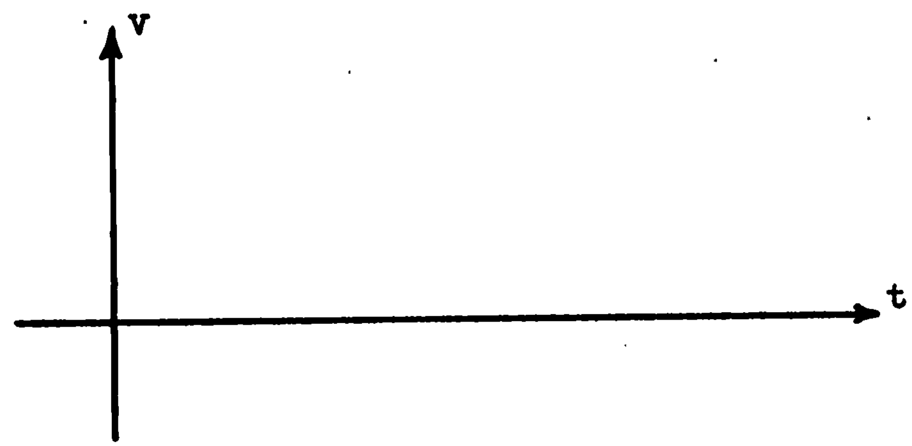


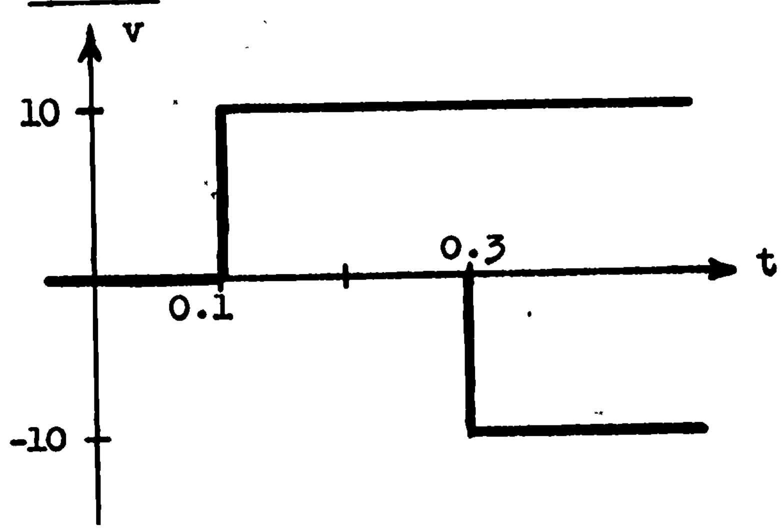
Fig. 16

Now consider the <u>pulse</u> having a 10-volt magnitude shown in Fig. 16. On the set of axes below show how two step functions can be combined graphically to yield the given pulse.



Also write a mathematical expression.

Answer:



Add the two step functions. Everywhere beyond $t = 0.3$ they cancel each other and leave only the pulse between 0.1 and 0.3. The mathematical expression is

$$v = 10\ u(t-0.1) - 10\ u(t-0.3)$$

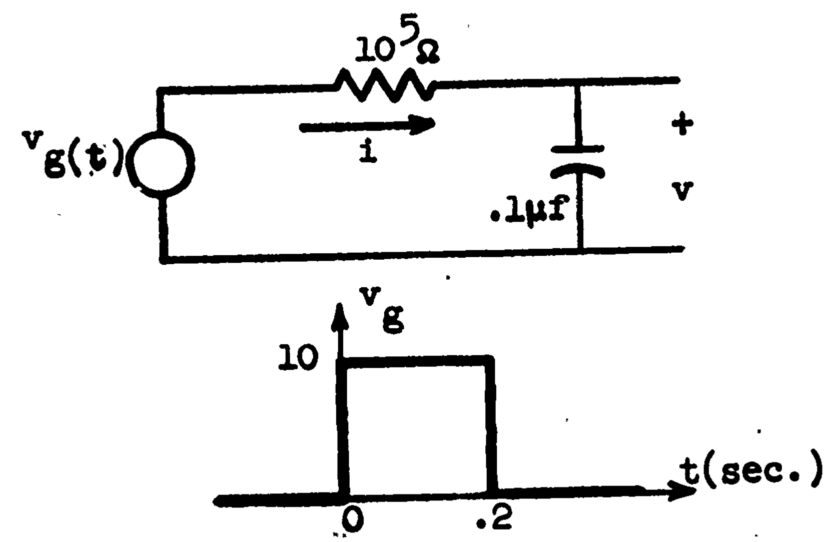


Fig. 17

Thus, if the source voltage in Fig. 9 is a pulse, we can find the capacitor voltage and current by using, twice, the previously obtained solution for a constant source voltage, once with a positive value, and once with a negative value but occurring at a later time.

In Fig. 17 suppose $v_g(t)$ is the pulse shown and suppose the capacitor is uncharged before the initiation of the pulse. It is desired to find the capacitor voltage v and its current i from t = 0 to 1 second.

There are two periods of time to consider; the time immediately following each of the two step functions. For each time period the curve of v or i against time can be sketched when three quantities are known. These three quantities are:

a) ______________________

b) ______________________

c) ______________________

Answer:

a) the initial value
b) the ultimate, or final, value
c) the time constant

Remark

The ultimate value is the value which the variable in question (v or i) would reach if the state of the network did not change. The network cannot anticipate a change which will take place at a later time. Thus, after the pulse is initiated and before it is terminated, the network will behave as if the step which initiated the pulse will remain on.

Let's take the first time period (when the pulse is on). Specify, in the table below, the appropriate values of time constant, initial value and final value of the capacitor voltage and current

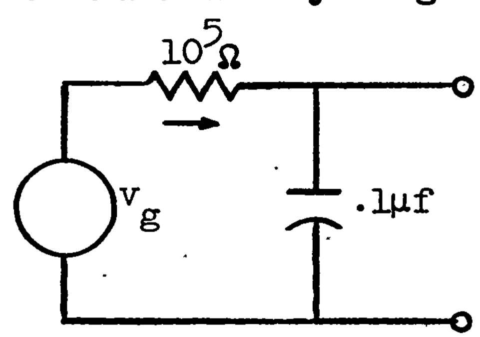
	voltage v	current i
time constant	____sec.	____sec.
initial value	____volts	____ma.
ultimate value	____volts	____ma.

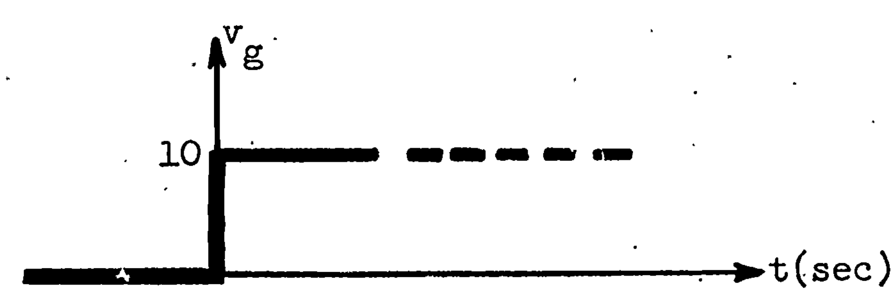
Hint: Verify that Kvl is satisfied by the initial values and final values in your answer.

Answer: FOR THE FIRST TIME PERIOD

	voltage v	current i
time constant	.01 sec.	.01 sec.
initial value	0 volts	.1 ma.
ultimate value	10 volts	0 ma.

Note that $10^5 i + v$ must equal 10 volts at all times after the pulse is initiated. The ultimate values given here are the values that would exist if the pulse were to remain on for a very long time compared to the time constant.





Let's continue this analysis of the first time period.

Now write the general expressions for the capacitor voltage **and** current as they apply to this time period, using the values just computed to evaluate all unknown constants.

$v(t) =$ ________________ volts $0 < t < 0.2$

$i(t) =$ ________________ amps. $0 < t < 0.2$

<u>Answer</u>:

$$v(t) = \underline{10(1-e^{-100t})} \text{ volts}$$

$$i(t) = \underline{10^{-4}e^{-100t}} \text{ amps.}$$

Solution for v(t):

$$10^5 i + v = 10 \;;\quad 10^5(.1\times10^6)\frac{dv}{dt} + v = 10 \;;\quad \frac{dv}{dt} = 1000 - 100v$$

$$\frac{dv}{1000-100v} = dt \;;\quad -\frac{1}{100}\ \ell n(1000-100v) = t + K_1 \;;\quad 1000 - 100v = k_2 e^{-100t}$$

$$\text{at } t = 0,\ v = 0 \;;\quad K_2 = 1000 \;;\quad v = 10(1 - e^{-100t})$$

The duration, or length, of the pulse is 0.2 second which is a factor of ________ times the time constant. What can you then conclude concerning the degree to which the capacitor voltage and current have reached their ultimate values by the end of the pulse?

Compute the value of v and i just at the end of the pulse to verify your conclusion.

Answer:

20

Since the capacitor voltage and current reach a value within 2 per cent of their final value in 4 time constants, in 20 time constants they will have reached their ultimate value almost completely.

$$v(0.2) = 10(1-e^{-20}) = 10(1-2\times10^{-8}) \doteq 10$$

$$i(0.2) = 10^{-4}e^{-20} = 10^{-4}(2\times10^{-8}) \doteq 0$$

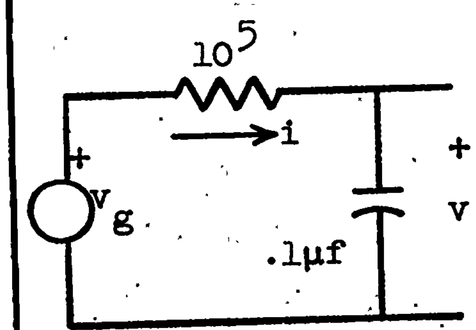


Fig. 17
(repeated)

Now let's turn to the second time period (after the termination of the pulse). The <u>initial</u> time for this period is the end of the pulse. Complete the table below for the capacitor voltage and current.

	voltage v	current i
time constant	____ sec.	____ sec.
initial value	____ volts	____ ma.
ultimate value	____ volts	____ ma.

(Be careful of references, and make sure your answers are consistent with Kvl.)

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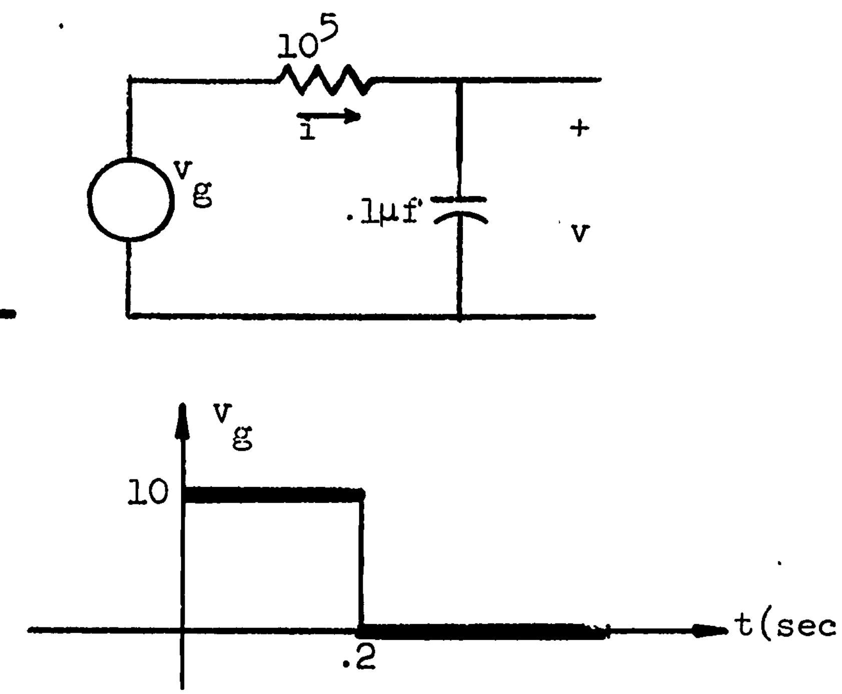
Answer:

	voltage v	current i
time constant	.01 sec.	.01 sec.
initial value	10 volts	-.1 ma.
ultimate value	0 volts	0 ma.

Note that the current has now reversed.



It is desired now to write expressions for the voltage and current as a function of time during the period following the end of the pulse. We know the initial values of v and i at the start of this interval. The situation can be considered as the start of a new problem. It is convenient to consider the starting point as the zero of time. For this reason we can define a new time variable t' which is 0.2 sec. less than the old time variable t; that is, $t' = t - 0.2$. Thus, at the end of the pulse, when $t = 0.2$, the new time is $t' = 0$.

Write expressions for v and i in terms of the time variable t' using the values from the previous page. Then substitute for t' in terms of t.

$v(t') =$ ______________ volts. $t' > 0$

$i(t') =$ ______________ amps. $t' > 0$

or

$v(t) =$ ______________ volts $t > 0.2$

$i(t) =$ ______________ amps. $t > 0.2$

<u>Answer</u>:

$v(t') = \underline{10e^{-100t'}}$ volts $\qquad v(t) = 10e^{-100(t-0.2)} \qquad t > 0.2$

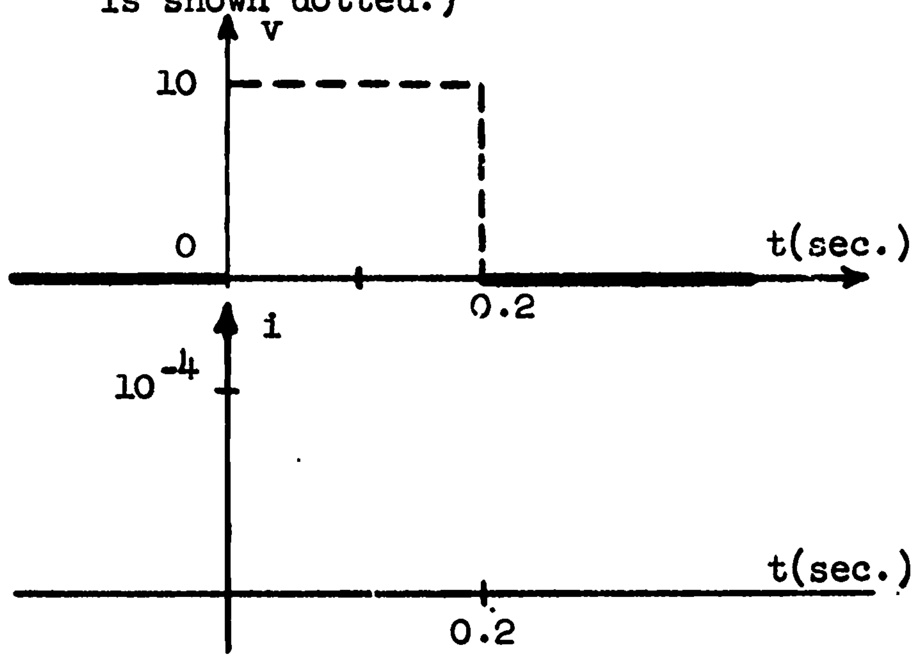
$i(t') = \underline{-10^{-4}e^{-100t'}}$ amps. $\qquad i(t) = -10^{-4}e^{-100(t-0.2)} \qquad t > 0.2$

These are to be compared with the previously found expressions (repeated below) for the period when the pulse is on.

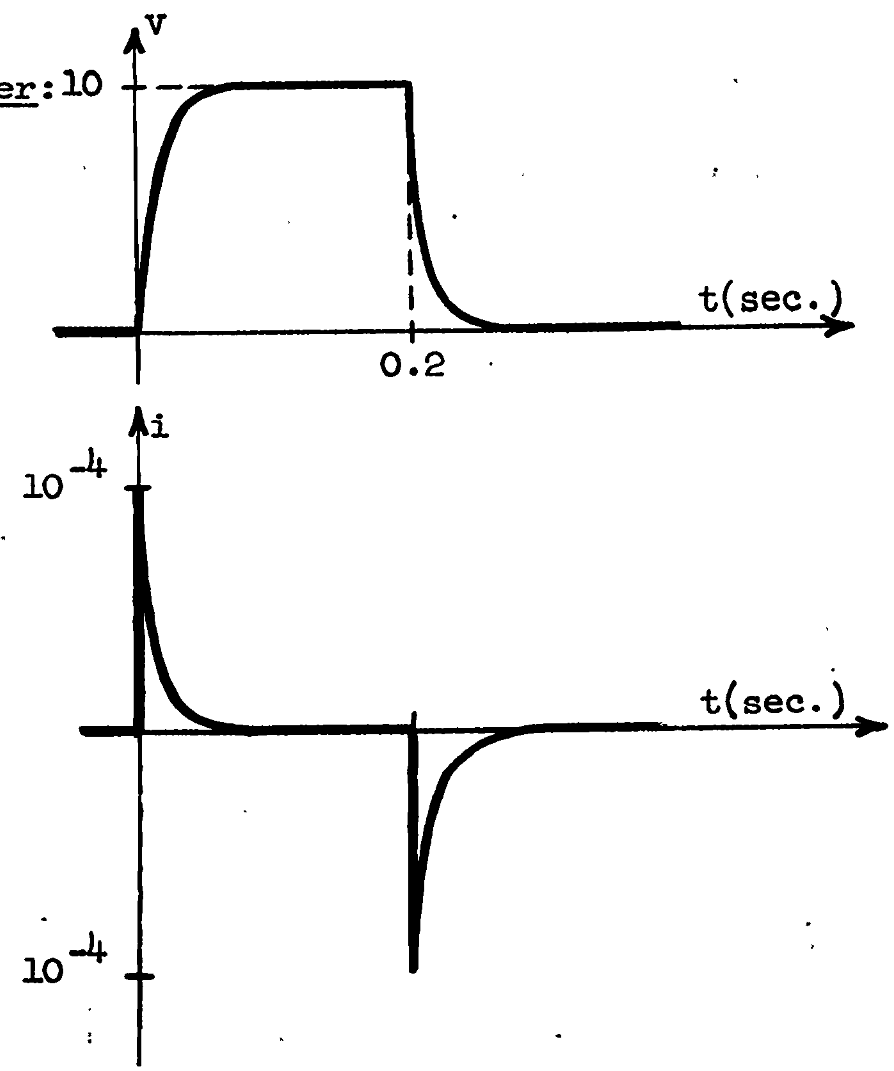
$$v(t) = 10(1-e^{-100t}) \qquad 0 < t < 0.2$$

$$i(t) = 10^{-4}e^{-100t} \qquad 0 < t < 0.2$$

Finally, make a careful sketch on the axes below of the voltage and current waveshapes as a function of time, from slightly before the initiation of the pulse to a time equal to twice the pulse duration. (The outline of the applied pulse is shown dotted.)



Answer:



Notice the shape of these curves. The capacitor voltage still looks like a pulse, but with somewhat rounded corners. The current consists of two spikes, the maximum current occurring when there is a change in the input voltage. As the capacitor voltage increases toward the pulse voltage, the voltage across the resistor (and hence also the current) decreases, in accordance with Kirchhoff's voltage law.

The degree of rounding of the corners of the pulse will depend on the ________ ________. For the case under consideration, the time constant is ____________ (how many?) times the pulse width. In this case we find that the applied voltage pulse is transmitted to the output with a small amount of distortion.

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Answer:

time constant

1/20, or 0.05

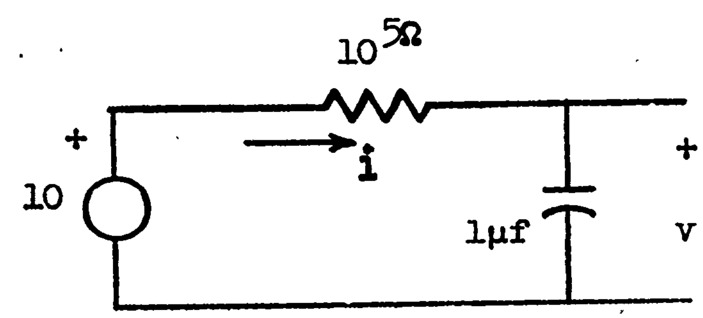
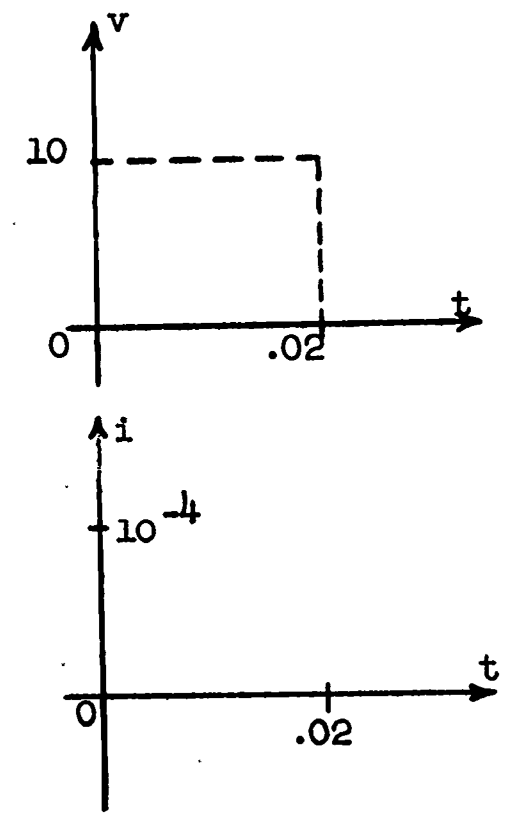


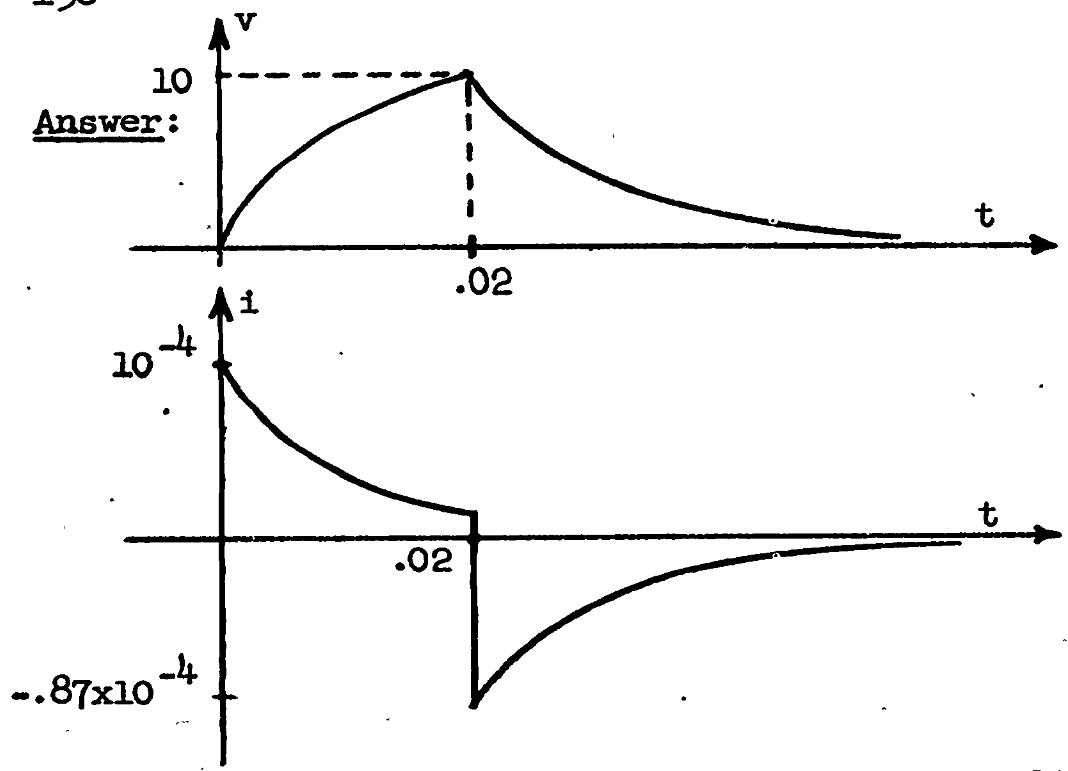
Fig. 18



Let's consider changing the time constant to observe what influence it has on the waveshape of the capacitor voltage and current. Suppose that the capacitance in Fig. 17 is changed to 1μf, everything else remaining the same. Sketch the resulting curves of $v(t)$ and $i(t)$. Also, write the corresponding expressions for $v(t)$ and $i(t)$ in the two time intervals. (You may have some difficulty with the second interval. Note that it starts at $t = .2$ and so your expression should account for this.)

$\underline{0 < t < 0.2}$	$\underline{t > 0.2}$
$v(t) =$ ______	$v(t) =$ ______
$i(t) =$ ______	$i(t) =$ ______

Answer:



$\underline{0 \leq t \leq 0.2}$

$v(t) = 10(1-e^{-10t})$

$i(t) = 10^{-4}e^{-10t}$

$\underline{t \geq 0.2}$

$v(t) = 8.7e^{-10(t-.2)}$

$i(t) = .87\times10^{-4}e^{-10(t-.2)}$

Notice how much more distorted the capacitor voltage is now compared to what it was when the time constant was smaller.

Now consider the network shown in Fig. 19. The switch has been closed for a long time so that the network is in equilibrium. At a time which we can take to be $t = 0$, the switch is opened. Ten μsec. later it is again closed. Sketch the waveshapes of $v(t)$, $i_1(t)$ and $i_2(t)$ approximately to scale from $t = -10$ to $t = 40$ μsec. Label the axes.

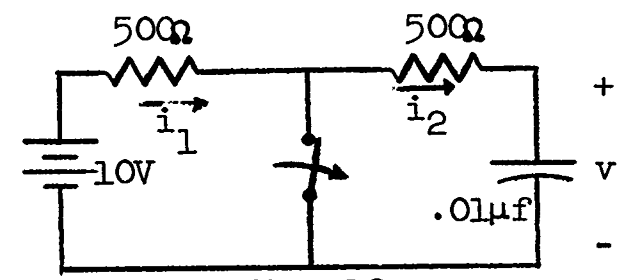
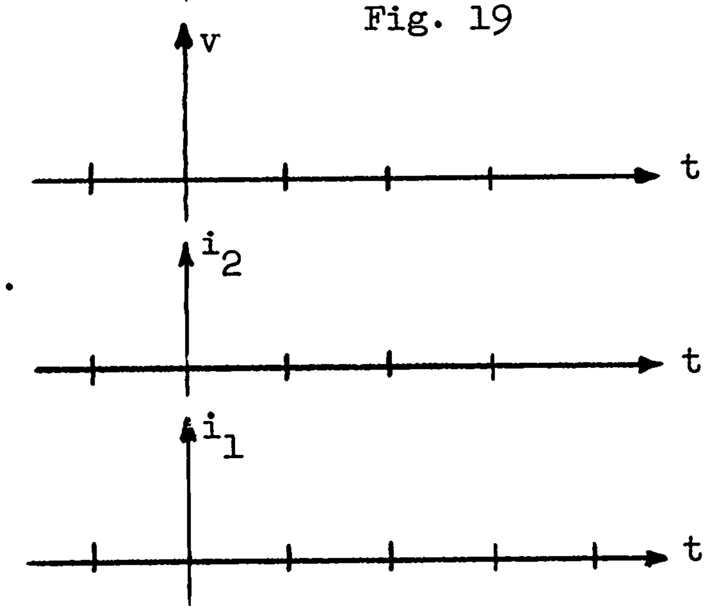
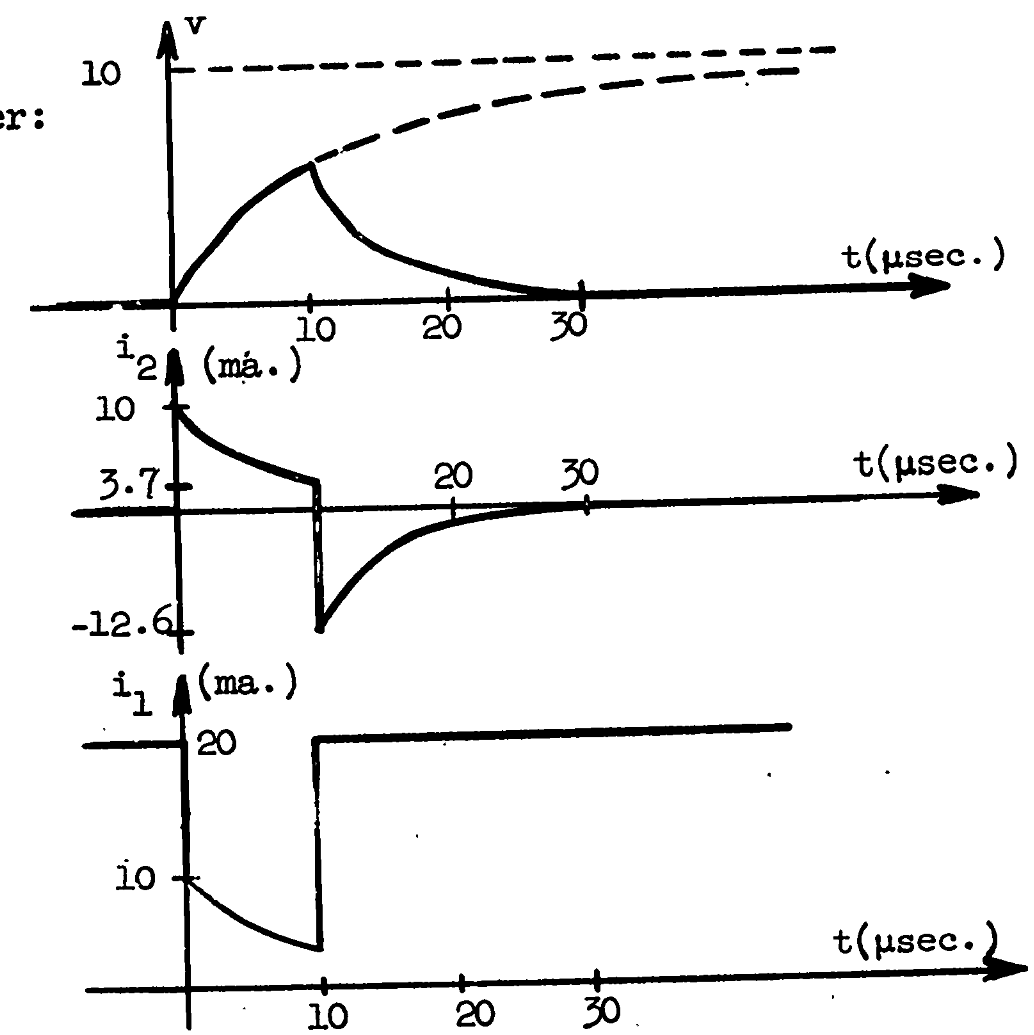


Fig. 19



160

Answer:



v
10
t(μsec.)
10
20
30
i_2 (ma.)
10
3.7
-12.6
t(μsec.)
20
30
i_1 (ma.)
20
10
t(μsec.)
10
20
30